Full research paper

ENHANCING PRE-SERVICE MATHEMATICS TEACHERS' PROOF-WRITING SKILLS: THE EFFECT OF A SOCIAL LEARNING ENVIRONMENT ENRICHED WITH DYNAMIC GEOMETRY SOFTWARE

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ABSTRACT

Mathematical proof, often regarded as the heart of mathematics, is essential for interconnected mathematical knowledge. However, proof-writing skills do not develop inherently. Effective learning environments are vital for university students to enhance these skills. This study investigates the impact of the ISMAT model on pre-service teachers' proof-writing skills. The model, based on quasi-experimental paradigms and arguments from Popper (1979) and Lakatos (1961, 1976), utilizes dynamic geometry software to enhance the understanding of proof functions. It is hypothesized that a social learning environment, augmented by dynamic geometry, will yield observable effects. The research employed a quasi-experimental design with experimental and control groups of pre-service mathematics teachers. The experimental group received 14 weeks of Euclidean geometry lessons using the ISMAT model, while the control group followed traditional methods. Data were collected through proof-writing tests administered pre- and post-instruction. The evaluations were conducted using Senk's (1983) framework for assessing proof-writing skills. Results indicated that the ISMAT model significantly enhanced proof-writing skills compared to traditional teaching methods. Such approaches are recommended to foster active student engagement in the proving process.

KEYWORDS

Instructional model, proof teaching, reasoning

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Highlights

- *The ISMAT Model underpins the design of proof-writing teaching environments.*
- *The model integrates social dimensions, proof functions, and reasoning types.*
- *The model is adaptable to various educational levels for proof-writing teaching.*
- *A social learning environment enhanced with dynamic geometry software effectively enhances proof-writing skills.*

INTRODUCTION

The advancement of mathematical knowledge requires geometry to improve deductive reasoning (González & Herbst, 2006). Proofs are crucial for conveying essential mathematical concepts (Hanna & Barbeau, 2010). Therefore, proofs are fundamental to the essence of mathematics (Ross, 1998), and the discipline is fundamentally about proof-writing (Heintz, 2000). As a result, the development of proof skills is crucial in Turkey's mathematics and geometry education, alongside skills such as data reasoning and problem-solving (MEB, 2013; Nasibov & Kaçar, 2005; Toluk, 2003). In school mathematics, the main goals of proof include fostering explanation, insight, and deep understanding (e.g., Dickerson & Doerr, 2014; Hanna, 2000; Stylianides, 2009; Yackel & Hanna, 2003). Thus, proof-writing is a fundamental aspect of mathematics teaching. Proof is essential in mathematics for disseminating and comprehending knowledge (Dolev & Elen, 2013; Ozturk, 2016; Mariotti, 2006). It serves to validate statements and reveal fundamental truths (Stylianou, Blanton & Knuth, 2009; Hanna, 2000). Additionally, proof incorporates a social aspect, involving mental activities like conjecturing and logical deduction (Hanna, 1991; Greenberg, 1993). Despite its significance in education, research shows students encounter notable challenges with mathematical proof at various educational levels (Heinze et al., 2008; Hoyles & Küchemann, 2002; Knuth, Choppin & Bieda, 2009; Stylianides & Stylianides, 2017). These challenges encompass a lack of initiation knowledge, misinterpretation of the necessity of techniques, and the perception of proofs as superfluous (Weber, 2001; Burgett, 2018; Buchbinder & McCrone, 2020). Furthermore, the perceived simplicity of proofs is exacerbated by non-collaborative activities, which are vital for the proving process (Cilli-Turner, 2017; Grundmeier et al., 2022; Melhuish et al., 2022; Yoo & Smith, 2007). Students' engagement with proof frequently remains static despite the implementation of diverse pedagogical approaches aimed at mitigating these challenges (Bermudez & Graysay, 2025; Ndemo, Mtetwa & Zindi, 2019).

This research focuses on the social dimensions inherent in the construction of mathematical knowledge, the functions of proof, and how a pedagogical approach involving the integration of dynamic geometry software into the learning environment affects pre-service teachers' skills in writing mathematical proofs.

Theoretical Background

Proof enhances mathematical knowledge and understanding (Hanna & Barbeau, 2010). Recognizing connections and comprehending mathematical thought are essential for proof (Flores, 2002). Forman et al. (1998) emphasize the importance of proof-writing in developing mathematical language. Proof distinguishes mathematical statements from other ideas, conferring significance (Hersh, 2009). However, proof involves exploration, conjecturing, reasoning, and argument formulation (Hanna et al., 2004; Pedemonte & Reid, 2011; Remillard, 2009; Stylianides & Ball, 2008). Researchers contend that proof fulfills various functions beyond verification in mathematics education (Almeida, 2003; Bell, 1976; De Villiers, 1990; Hanna & Jahnke, 1996; Hanna, Jahnke & Pulte, 2010). De Villiers (1990) builds on Bell's (1976) functions of proof, adding discovery and communication. The functions of proof encompass verification, explanation, systematization, discovery, and communication. De Villiers (1999) argues that proof presents an intellectual challenge, thus adding this function. Consequently, proof involves investigating mathematical phenomena through pattern recognition, conjecturing, and argumentation (Stylianides, 2008). In mathematics education, proof significantly contributes to the augmentation of mathematical proficiency and logical reasoning (Hanna, 2000).

Challenges in the Proving Process

Proof is fundamental for mathematical comprehension and communication (e.g., Ball & Bass, 2003; Hanna Larvor & Yan, 2023; Herbst & Brach, 2006). However, the significance of proof in fostering mathematical thought is often underestimated. Consequently, proof-writing presents challenges for students at all academic levels (e.g., Jones & Herbst, 2012; Stylianides & Stylianides, 2017; Stylianou, Blanton & Knuth, 2009). Numerous studies identify various factors contributing to difficulties in proof-writing (e.g., Chazan, 1993; Harel & Fuller, 2009; Harel & Sowder, 2007; Moore, 1994; Selden & Selden, 2007; Weber, 2006). Moore (1994) delineates key sources of students' challenges in proof-writing, including perceptions of mathematics and proof, conceptual understanding, mathematical language, and initiation of proof. Harel and Sowder (2007) emphasize the role of cognitive factors in shaping students' engagement with proof. They assert that multiple influences, such as students' or teachers' attitudes towards proof and the design of the learning environment, affect students' proof-related behaviors, necessitating a multifaceted approach to understanding their difficulties. This indicates that an array of factors impacts the proving process. Therefore, a thorough understanding of the obstacles in proof-writing is imperative. Moreover, it is crucial to create educational settings that facilitate proof-writing and to employ pedagogical strategies that reflect the intrinsic nature of proof and encourage student involvement in proof activities.

Proof-Writing Skill

Proof is a comprehensive process reliant on visual or experimental evidence, logical reasoning, and personal convictions (Hoyles & Healy, 2007). Proof-writing is not merely about demonstrating mathematical truth; it embodies a cognitive framework. This is supported by Ball et al. (2002), who argue that it involves cognitive habits like identifying constructs, exploring, formulating assumptions, and organizing reasoning. Greenberg (1993) further substantiates this by noting that writing proofs includes various cognitive activities such as generating assumptions and deriving logical conclusions. Thus, proof-writing is a significant task that requires a wide range of skills, including reasoning and problem-solving, which enhance mathematical cognition. Therefore, instructional methods that encompass diverse skills and recognize the essence of proof are linked to improved proof-writing skills among students. This is echoed by Senk (1983), who attributed proof-writing difficulties to the characteristics of the mathematical system, cognitive development stages, and the pedagogical approaches employed. Furthermore, Senk's consideration of reasoning, justification, and mathematical language in her research on proof-writing and geometry comprehension highlights the complexity inherent in the proving process. As such, the critical actions in writing proofs and the identification of instructional methods as factors in these challenges emphasize the necessity for learning environments that align with the intrinsic nature of the proving process.

In academic research examining proof-writing (e.g., Ko & Knuth, 2009; Moore, 2016; Senk, 1983; Stylianides & Stylianides, 2009; Winer & Battista, 2022), qualitative analyses focus on essential criteria for validating an argument as a legitimate proof. Notably, reasoning, mathematical language, and justification are critical factors in evaluating proofs in these investigations. The five-level scoring scale created by Senk (1983) systematically assesses proofs, covering these dimensions and providing scores for each criterion, thereby enhancing its value as an evaluative instrument. This framework increases its efficacy by promoting thorough and explicit evaluations of proofs. The scoring scale proposed by Senk (1983) is outlined below.

- 0 – Student writes nothing, writes only the given, or writes invalid or useless deductions;
- 1 – Student writes at least one valid deduction and gives a reason;
- 2 – Student shows evidence of using a chain of reasoning, either by deducing about half the proof and stopping, or by writing a "proof" that is invalid because it is based on faulty reasoning early in the steps;
- 3 – Student writes a proof in which all steps follow logically, but in which there are errors in notation, vocabulary, or names of theorems;
- 4 – Student writes a valid proof with at most one error in notation.

Instructional Approaches for Teaching Proof and Proposed Model

The primary benefit of proof in mathematics education is enhancing mathematical comprehension (Hanna & Jahnke, 1996; Hersh, 1997). Traditionally, proof serves to confirm the accuracy of mathematical statements (Avigad, 2005; De Villiers, 1990). Nevertheless, numerous researchers emphasize that proof encompasses various functions beyond mere validation in pedagogical contexts (e.g., Almeida, 2003; Bell, 1976; De Villiers, 1990; Hanna & Jahnke, 1996; Hanna, Jahnke & Pulte, 2010). These researchers identify analogous functions associated with the mathematical significance of proof. De Villiers (1999) elaborates on Bell's (1976) framework by incorporating discovery, communication, and intellectual challenge as additional dimensions of proof. Stylianides (2009) corroborates this diversity of functions, indicating that mathematical proof involves actions such as generalizing patterns, formulating conjectures, constructing arguments, assessing others' conjectures or arguments, and disseminating mathematical knowledge. Therefore, it is crucial to adopt proof teaching methods that align with the nature of proof and encourage student engagement in proof activities. Furthermore, it is imperative to foster learning environments conducive to proof teaching that enable students to engage with the procedural steps similar to those undertaken by mathematicians during the proving process.

In undergraduate mathematics, proof teaching follows a standard deductive sequence of definition, theorem, and proof (Almeida, 2000). Moreover, proofs are presented solely as final products, depriving students of practical proof experiences (Alibert & Thomas, 1991; Ferrari, 2004). Consequently, this methodology results in a deficient comprehension of mathematical proof among students (Knuth & Elliot, 1997). Therefore, pre-service mathematics teachers should be immersed in environments where the concept of proof is emphasized, enabling them to engage in the process of proving. Enhancing their skill in this area will empower them to mentor students based on personal experiences. Consequently, acknowledging the increasing importance of proof within the realm of mathematics and its educational practices, reformulating instructional methodologies is essential.

In the domain of proof teaching literature, numerous studies focus on the enhancement of proof-writing skills (Bobango, 1987; Cook-Box, 1996; Generazzo, 2011; Hart, 1986; Hsu, 2010; Lee, 1999; Lee, 2011; Matsuda, 2004; Pulley, 2010; Senk, 1983; Sommerhoff, Kollar & Ufer, 2021; Subramanian, 1991; Tubridy, 1992). Pulley (2010) specifically investigated how non-traditional instructional activities impact students' mathematical understanding, beliefs about proof, and reasoning. The study involved students in activities that required them to create, justify, and validate proofs, resulting in advancements in geometric knowledge and reasoning. In contrast, Generazzo (2011) assessed the effects of an inquiry-based learning environment on students' skills in conjecturing, reasoning, and proof-writing through collaborative group work and discussions. Sommerhoff, Kollar, and Ufer (2021) explored the effectiveness of sequential versus concurrent instructional methods on developing mathematical argumentation and proof skills, revealing that both strategies significantly enhance foundational resources for these skills, especially for lower-performing students. The findings emphasized that interactive student activities substantially improve proof-writing and reasoning skills. Additionally, it was observed that studies assessing the effectiveness of proof teaching practices are less prevalent compared to descriptive studies on proof. Furthermore, these studies (e.g., Marrades & Gutiérrez, 2000; Selden, Selden & McKee, 2008; Smith, 2006) aim to evaluate the impact of specific teaching methods or technologies on proof-writing. In studies that define multiple steps for proof-writing, these steps are employed to facilitate various activities; however, they lack the comprehensiveness of the instructional models utilized in proof teaching. This research holds significance for three primary reasons: it offers insights for developing teaching interventions to enhance students' proof-writing skills and address cognitive challenges, it serves as a foundation for reforming proof teaching practices, and it aids in recognizing the diverse functions of proof and reshaping perceptions thereof.

Overall, the findings from these investigations suggest that alternative pedagogical approaches yield advantageous outcomes for students in the domain of proof-writing. Furthermore, they emphasize the critical role of delineating instructional models that enable learners to engage actively in the proving process. In response to the stated imperative, this study aims to propose a conceptual model that aspires to demonstrate that the proving process extends beyond the mere validation of a particular theorem; it also functions to elevate this endeavor as a substantial intellectual pursuit from the learners' perspective by promoting their involvement with the essential activities that are foundational to the nature of proof. The instructional model aims to reveal that proof-writing is not just an action to demonstrate the accuracy of a given theorem, but also to make it a meaningful process for the students by allowing them to experience the inherent stages of proof-writing. The Model includes seven stages: *understanding the problem, constructing a structure, working on the structure and conjecturing, postulation of the relationship, proving, investigating the coherence of the proof,* and *formalizing the proof.* The realization of each of the reasoning types (deduction, induction, abduction), the reflection of the functions of proof (discovery, verification, explanation, systematization, communication, mental challenge), the inclusion of the views of Popper (1979) and Lakatos (1961, 1976) on the knowledge formation process, and the importance of the social dimension in this process were all considered when developing the ISMAT Model stages. Popper stated that science operates in four steps: formulating a hypothesis, deducing observable and testable conclusions, testing those conclusions, and determining whether to accept or reject the proposition (Hodson, 2008). Lakatos characterised a scientific research programme as progressive and asserted that successive steps involve making testable predictions and confirming them (Worrall, 2003). This research program, especially in the context of Lakatos, was developed as a synthesis of Kuhn's and Popper's opposing views. In his book the Structure of Scientific Revolutions, Thomas Kuhn—the most significant historian and philosopher of science—presented a radically different view of science. With his book, he attempted to make sense of the assertion that scientists working under conflicting paradigms "*live in different worlds*."

Additionally, Popper (1979) emphasizes that individuals live in a mathematical world and asserts the existence of three different worlds. These three worlds are the physical, mental, and social worlds in that order. In the mental world, knowledge is derived from the individual's experiences and beliefs. It is determined in the physical world whether the subjective knowledge is applicable and if it validates individuals' experiences. In the mathematical world, individuals share their knowledge, and once that knowledge is confirmed, it becomes objective and universally accepted. Therefore, Popper and Kuhn advocate the presence of different worlds. According to Lakatos, Popper's third realm is where knowledge grows and is restructured, which highlights the same argument (Ozturk, 2016). The actions in the three worlds are compatible with the steps of the proving process. Scientists also begin their endeavours by thinking, speculating, and generating a new claim. After that, they support their arguments and produce new knowledge. Proof-writing is, obviously, a process of forming knowledge. As a result, the ISMAT Model combines the ideas of Lakatos, Popper, and Kuhn. Each of these researchers made a substantial contribution to the philosophy of science. The Model reflects how science philosophy is applied to mathematics. In other words, the Model is a reflection of the process of doing science.

In light of Stylianides's (2007) definition of the proof, Conner and Krejci (2022) assert that reasons, generality, clarity, and structure are the four essential components of proof. Since the Model includes these essential components, it serves as a representation of the proving process. Furthermore, the Model was designed to allow for both individual and group work, as well as the usage of dynamic geometry software. Nonetheless, within the parameters of the proposed model, the reciprocal exchange of proof drafts among the various groups, along with the provision for offering suggestions for amendments to the proofs, followed by a whole-class discussion regarding the proofs, is designed to facilitate the alleviation of the challenges encountered in writing proof. The stages and main principles of the ISMAT Model are presented in Figure 1.

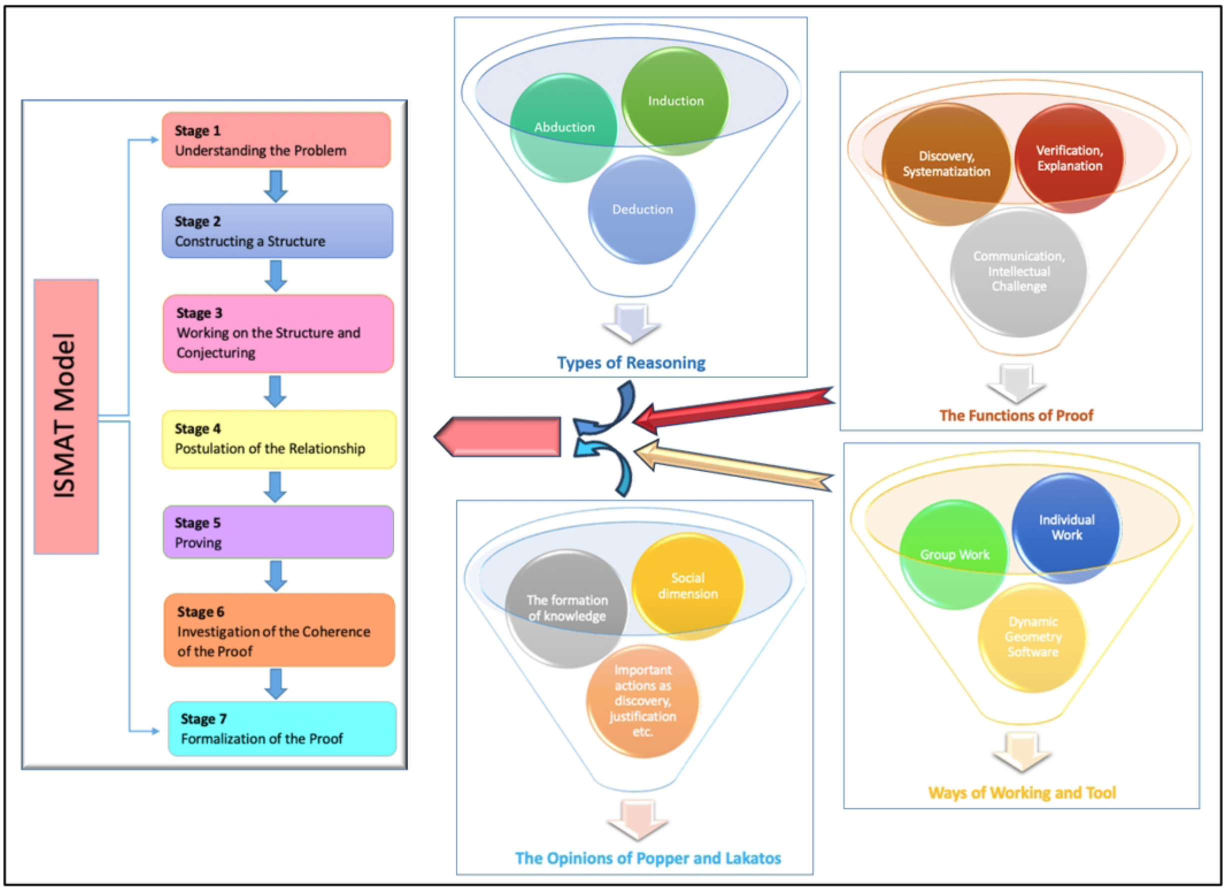


Figure 1: Stages and main principles of the ISMAT model (Ozturk, 2016)

The Purpose of the Study

Research (e.g., Moore, 1994; Stylianides, Stylianides & Shilling-Traina, 2013; Weber, 2001, 2006) indicates that students encounter considerable challenges in writing mathematical proofs in steps, even at the university level. Students exhibit notable deficiencies, particularly in employing logical connections among the steps of proof and utilizing mathematical language (Ko & Knuth, 2009; Miyazaki, Fujita, & Jones, 2016). Indeed, these deficiencies have been informally noted by the researchers of this study over an extended period, as they have been facilitating Euclidean geometry courses for numerous years. These insights significantly motivated the researchers to execute the ISMAT Model within an authentic classroom setting and to assess its outcomes. This study aimed to examine the effect of the ISMAT Model on pre-service teachers' proof-writing skills.

METHODOLOGY

Research Design

This study investigates the impact of the designed learning environment on pre-service mathematics teachers' proof-writing skills, assessing these skills both pre- and post-instruction, and analyzing the relationship between potential skill changes and the proposed model. An experimental approach was thus employed. At this point, a quasi-experimental study comparing two groups where only one is exposed to the ISMAT Model would be somewhat explanatory in terms of the proof-writing skills. However, controlling all variables except the instructional model for more than one group was quite difficult, making it challenging to draw inferences about the reasons for the change. Besides, the characteristics of the course were a threat to a quasi-experimental design due to the challenges, such as restricting communication between groups. By the way, we could apply interviews through the change, which involves obtaining quantitative data, and try to uncover the underlying reason for the change.

Participants

Every year in Turkey, students at all K–12 levels are introduced to geometry concepts before attending university. Examining the characteristics of geometric shapes and objects in middle school replaces traditional geometry teachings, which focus on identifying geometric shapes and objects, particularly in the early years of education, known as the primary school years. In the last four years, also known as high school, in addition to investigating the properties of geometric shapes and objects, elements such as proofs of basic relations and geometric drawings are included. In geometry courses, students are given exercises and problems that require the application of geometric object properties, as the Turkish educational system employs a centralized exam system for admission to reputable high schools and universities. Students cannot experience formal proof-writing activities in these courses, which are structured by centralized assessments. Students are first exposed to proofs in a formal sense in university mathematics and teaching programs, which are typically entered through central exams. As a result of the central exam, the study's sample consists of first-year students attending the mathematics teaching program at a university with a medium level of success in the central exam.

The experimental implementation of the proposed model was conducted within the scope of the "Euclidean Geometry" course, taught in the first year of the program. In this context, the study sample consisted of a total of 60 pre-service mathematics teachers, divided into two groups: 32 in the experimental group (27 girls and five boys) and 28 in the control group (15 girls and 13 boys).

This retrospective research involving human participants was in accordance with the ethical standards of the institutional and national research committees. The Social and Human Sciences Ethics Committee of Karadeniz Technical University approved this research. (Ref. No. 82554930/400-1259)

Data Collection Tool

The data were collected through a "proof-writing test" (PWT). However, upon examining the proof-writing tests, certain responses from pre-service teachers were ambiguous and lacked clarity, prompting subsequent interviews for further elucidation of their answers.

Each interview was conducted with a cohort of six pre-service teachers. The selection process for the pre-service teachers participating in the interviews was predicated on pre-test outcomes and the principle of voluntary participation. The duration of each interview was approximately 40 minutes, conducted in a setting that fostered comfort for each pre-service teacher, and incorporated various proof-writing activities. Furthermore, the interviews were administered on an individual basis to mitigate the potential for inter-participant influence among the pre-service teachers.

Proof-Writing Test

Two separate proof-writing tests were administered as both a pre-test and a post-test, preceding and succeeding the experimental intervention, to evaluate the proficiency of pre-service teachers in the domain of proof writing. Pre-service teachers were administered the Proof Writing Pre-Test (PWPRE), which comprises 12 questions, to assess their initial skills in proof-writing before the intervention. They were subsequently administered the Proof Writing Post-Test (PWPOST), also comprising 12 questions, to evaluate their proficiency following the intervention. The proof-writing questions contained within the PWPRE were meticulously designed to gauge students' existing proficiency before the intervention. They were intended to be addressed employing the foundational geometry knowledge they had amassed during their high school education. In contrast, the proof-writing questions of the PWPOST, aimed at evaluating the students' advancement post-intervention, were formulated such that they could be addressed utilizing both their high school knowledge and the newly acquired information from the Euclidean geometry course.

The initial phase in formulating the PWPRE and PWPOST tests involved selecting pertinent questions sourced from both university-level and high school textbooks, as well as from existing literature. In the process of selecting questions, consideration was given to the general high school curriculum for the pre-test and the content of the Euclidean Geometry course for the post-test, with a concerted effort made to ensure a broad scope in the coverage of the questions. To facilitate the preparation of the tests, two researchers holding doctoral degrees in mathematics education were solicited for their expert recommendations. Following revisions to the questions based on their feedback, a cohort of 45 pre-service teachers participated in a pilot implementation to assess the reception of the tests by the pre-service teacher population. Adjustments were made, and the questions were refined as a direct consequence of the pilot implementation. Following the pilot implementation, it was determined that one of the problems included in the pre-test should be substituted with an alternative problem, as it was perceived to be overly challenging for pre-service teachers who had only recently graduated from high school, thereby failing to fulfill the research objectives. In the post-test, it was considered judicious to eliminate the problem due to its requirement for direct engagement with the mathematical relationship delineated in the problems, which did not necessitate a sequence of proof steps. Nevertheless, it was deemed beneficial to incorporate a different problem into the post-test. Upon identifying that the duration allocated for completing these assessments was inadequate in the pilot implementation, it was resolved that the examination time for both assessments would be extended to 120 minutes. Figure 2 delineates the development process of the tests.

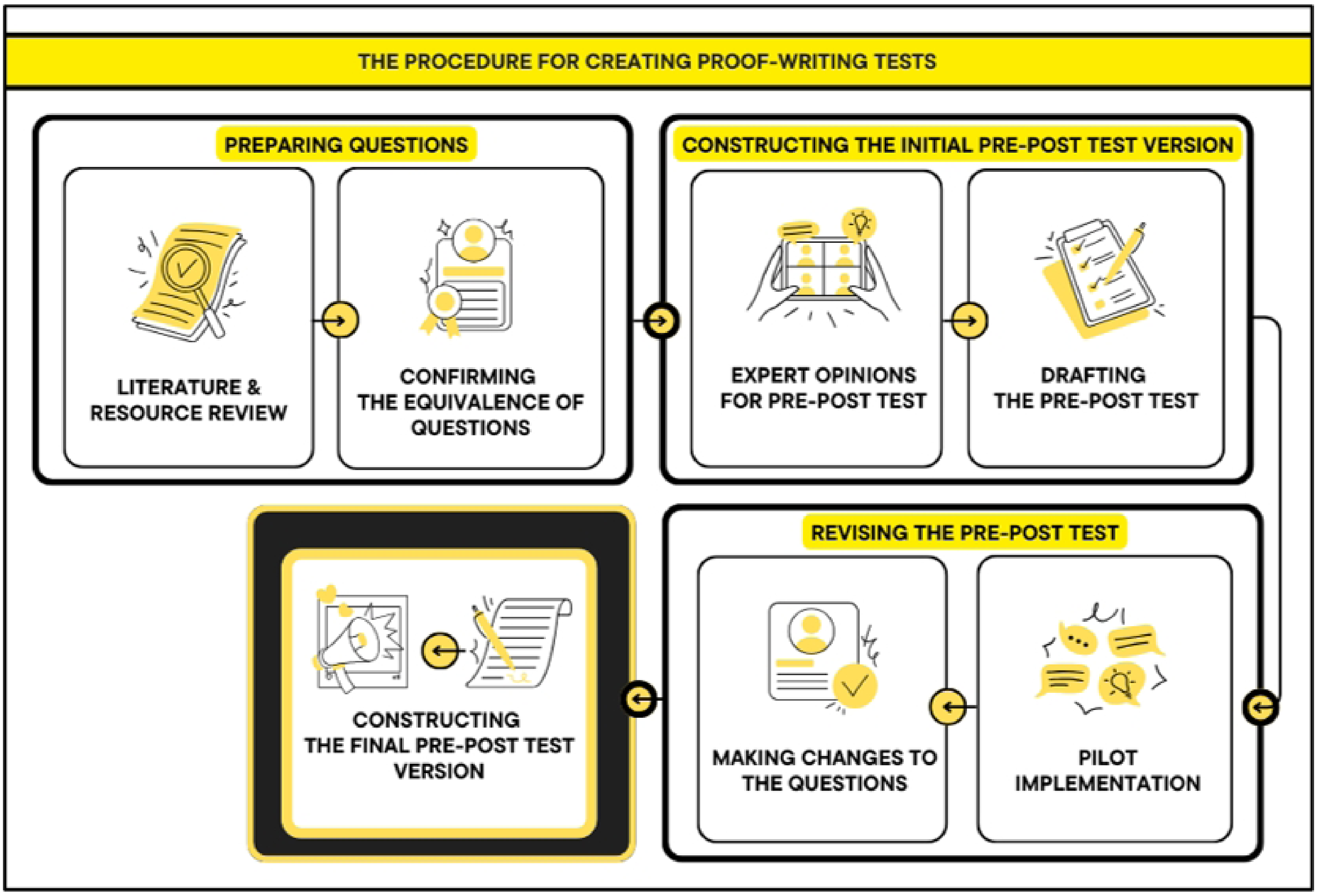
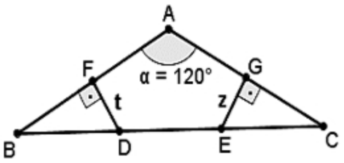
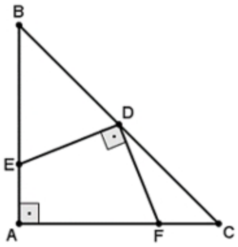


Figure 2: Procedure for creating proof-writing tests

Table 1 presents the detailed content of the questions in the PWPRE and PWPOST, as well as the prior knowledge required to demonstrate the mathematical relationships specified in the questions.

Test	Geometric Figure	Content	Prior Knowledge*
PWPRE		ABC is an isosceles triangle, $m(\hat{A}) = 120^0$. Given that the straight lines t and z are the perpendicular bisectors of [AB] and [AC] respectively, and that $t \cap [BC] = \{D\}$, $z \cap [BC] = \{E\}$, prove that $\lvert BD \rvert = \lvert DE \rvert = \lvert EC \rvert$ with your justifications.	The definition of an isosceles triangle The properties concerning secondary elements.
PWPOST		The midpoint of the hypotenuse of the isosceles right triangle ABC is D. Points E and F are located on the sides [AB] and [AC], respectively, such that $m(\widehat{EDF}) = 90^0$. Prove that $A(ABC) = 2.A(AEDF)$ with your justifications.	The definition and properties of an isosceles triangle The angle-side-angle congruence theorem.

**The knowledge that pre-service teachers can be used for proof-writing the problems.*

Table 1: Explanations for some questions in the PWPRE and PWPOST

Process

The implementation occurred within a geometry course. Certain geometrical topics were covered during this course. Weekly proof-writing activities were conducted for the experimental group. These activities employed the ISMAT Model through collaborative group work. Each group was composed of three pre-service teachers. The control group's practices varied in that they did not incorporate proof-writing activities. Instead, direct mathematical statements were provided for proof-writing. However, classroom discussions focused on the proofs, with the instructor as the sole user of the dynamic software. Figure 3 illustrates the implementation process of a proof-writing activity. At the conclusion of the implementation process, a post-test was administered to assess the pre-service teachers' proof-writing skills. The treatment duration for each group spanned 14 weeks, encompassing both the proof-writing assessments and the introduction of the GeoGebra software. Subsequently, the data were analyzed comprehensively and interactively. Table 2 serves as an illustrative example demonstrating the implementation process of a proof-writing activity accompanied by comprehensive explanations of the respective stages.

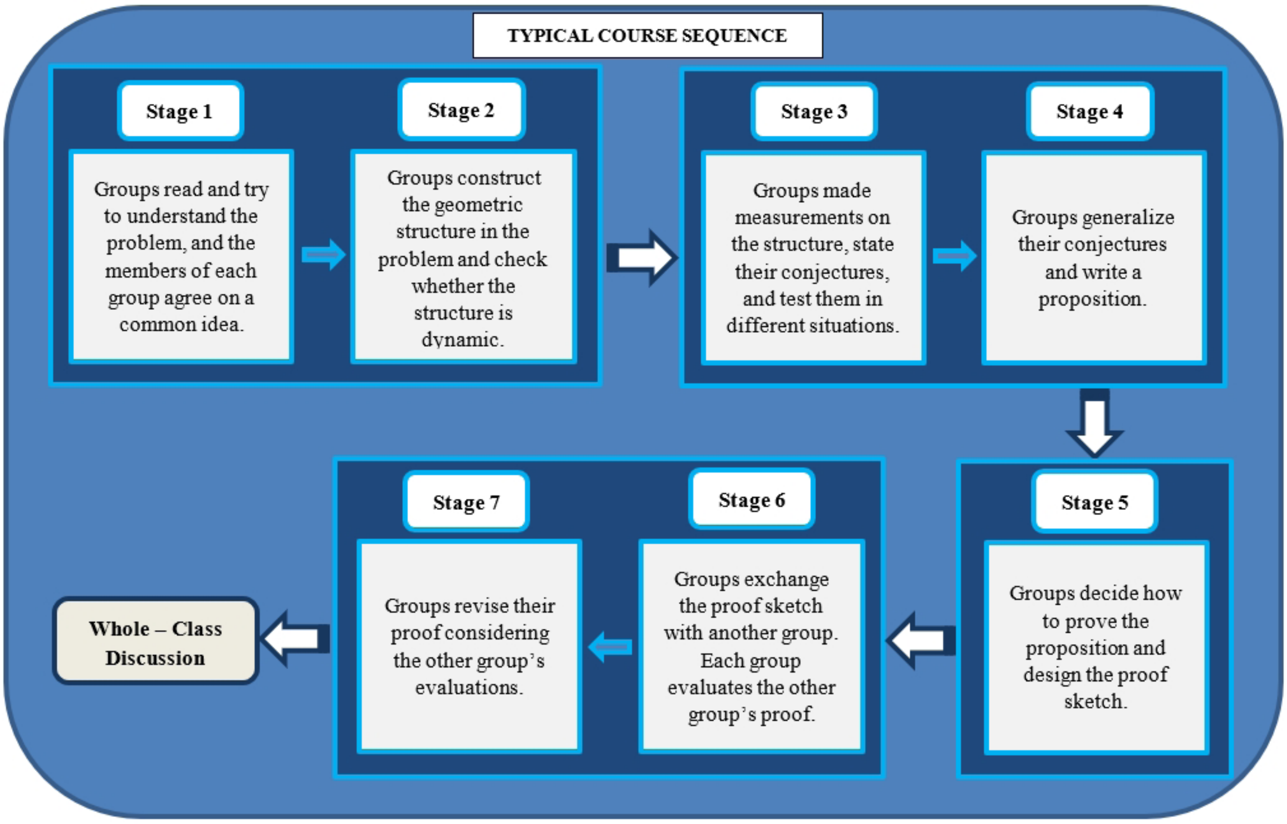
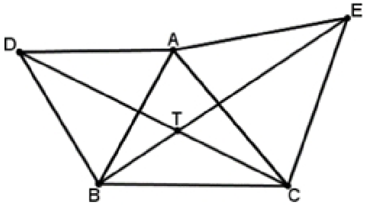
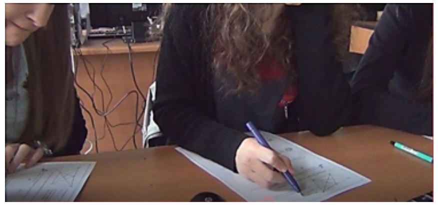
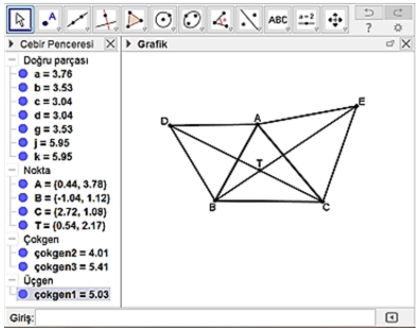
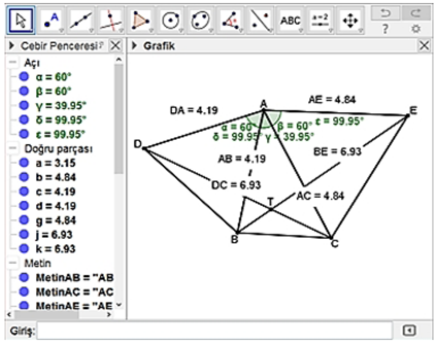
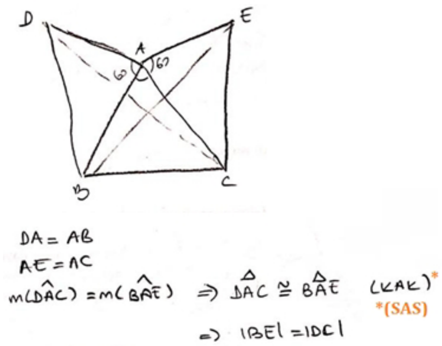
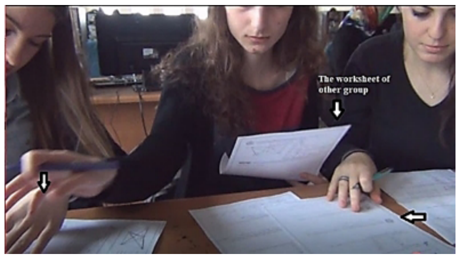


Figure 3: Typical course sequence based on the ISMAT model

<table>
<tr><th>Stage No</th><th>Description</th><th>Task</th><th>Reflection</th><th>Mathematical Behavior</th></tr>
<tr><td>1</td><td>Promote understanding of the problem by focusing on its essence, in other words, the givens.
Supporting the initial attempt to identify the existing mathematical relations.</td><td>

ABC is a triangle, BDA and AEC are equilateral triangles, $BE \cap CD = \{T\}$
According to the given information, determine whether there is a mathematical relationship between |BE| and |CD| by following the steps outlined below.</td><td></td><td>• Each group member reads about the problem.
• They try to understand the problem.
• Group members agree on a common idea.</td></tr>
<tr><td>2</td><td>Promote the construction of a structure by enabling one to think about the necessary geometric knowledge and drawing rules.</td><td>Considering the information provided, construct the geometric structure above.</td><td>
</td><td>• The group constructs the geometric structure in the problem by considering geometric knowledge and drawing rules.
• The group checks whether the structure is dynamic.</td></tr>
<tr><td>3</td><td>Promote working on the structure and conjecturing by enabling the making of numerous measurements on the geometric structure, searching for the mathematical relations, and testing the functionality of the conjectures for different situations.</td><td>a) Measure the lengths of the line segments BE and CD, and fill in the table with the corresponding measurement results. Note: The number of rows can be increased.
<table><tr><th>|BE|</th><th>|CD|</th></tr><tr><td></td><td></td></tr><tr><td></td><td></td></tr><tr><td></td><td></td></tr></table>b) According to your measurements and investigations, what is the mathematical relation between |BE| and |CD|? Write this mathematical relation below.
Would your conjecture be valid if any regular polygons were substituted for equilateral triangles? Note: State which regular polygons you use.</td><td>
</td><td>• The group conducts several measurements on the dynamic structure.
• The group searches for relationships in the structure based on the measurements.
• Each group member makes several conjectures based on these relationships.
• The group discusses and evaluates the applicability of its conjectures in various circumstances.
• At the end of their investigation and discussions, the group members state a common conjecture.
• The group tries to determine the steps of proof based on experimental data.</td></tr>
</table>

Stage No	Description	Task	Reflection	Mathematical Behavior
4	Promote the postulation of the relationship by providing an opportunity to determine the hypothesis and draw conclusions.	Based on your investigations, state your hypothesis and conclusion, and write the mathematical proposition by making use of them. **Hypothesis:** **Conclusion:**		• The group generalizes their conjectures. • The group writes a proposition. • The group uses mathematical symbols and terms during the postulation.
5	Promote proving by enabling the determination of the definitions, axioms, or theorems for showing the correctness of the proposition, and designing the proof plan.	Design a proof sketch and state the steps you will follow, along with their justifications.		• The group decides on a method to prove the proposition. • The group determines the definitions, axioms, and theorems that will be used in the proof. • The group designs the proof sketch.
6	Promote the investigation of proof coherence by allowing the exchange of proof plans among groups and evaluating the proof of one group by another.	Exchange your proof plans with another group and evaluate the group's proof. **Note:** *Write your evaluations of the proof below*. **The names of the other group members:** **The evaluations of the proof:**		• The group exchanges proof plans with another group. • Each group evaluates the proof of the other group. • The group determines any deficiencies and errors.

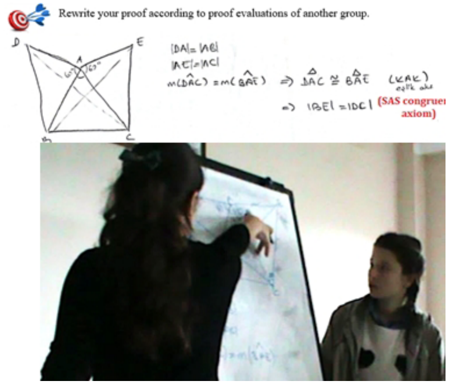
Stage No	Description	Task	Reflection	Mathematical Behavior
7	Promote formalization of the proof by providing an opportunity to revise deficiencies or errors in the proof plan.	Rewrite your proof according to the proof evaluations of another group.		• The group revises its proof in consideration of the other group's evaluations. • A group presents its proof to all groups. • A whole-class discussion is conducted. • The group finalizes its proof in accordance with the discussion that has taken place.

Table 2: Implementation process of a proof-writing activity based on the ISMAT Model

Data Analysis

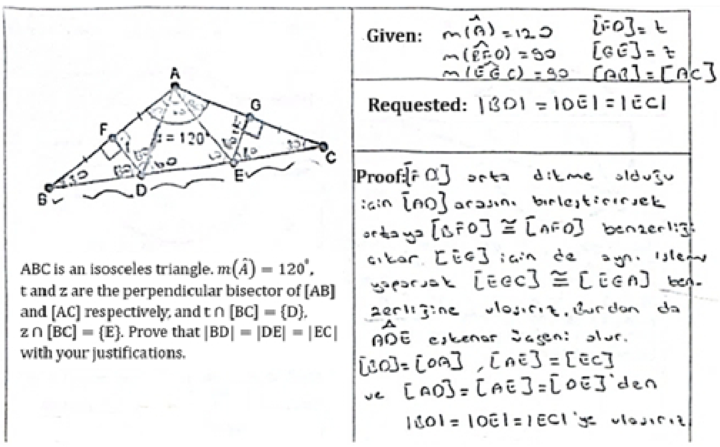
A modified version of Senk's (1983) 5-level scoring chart was utilized to assess data from proof-writing tests. This chart encompasses reasoning and mathematical language dimensions, allowing for integrated evaluations. While proof-writing entails various skills, including reasoning and justification, we posited that isolating these dimensions would yield more precise evaluations. Accordingly, *the Reasoning Process* was determined as one of the dimensions of the re-created scoring chart. Table 3 presents the categorical scoring chart formed to determine the pre-service teachers' proof-writing skills.

Reasoning Process (RP)	0- S/he left it blank. S/he wrote the hypothesis and conclusion in detail. S/he came up with irrelevant statements or inferences that did not contribute to the proof.
	1- S/he made at least one correct independent inference. S/he made his/her inference based on a case. However, while doing this, s/he was unable to provide sufficient justification. S/he made at least one inference starting from the conclusion.
	2- S/he made successive inferences supporting one another. However, s/he failed to attain the result. S/he attained the result through successive inferences based on a special case. S/he attained the result, but s/he did not formally justify the steps s/he took in the process of reaching it, or just provided an incorrect justification.
	3- S/he attained the result, but while some of the proof-writing steps were justified, others were not. S/he attained the result and justified a considerable part of the proof-writing steps. However, s/he made mistakes in some words and names of some theorems.
	4- S/he attained the result by justifying each proof-writing step.

Table 3: Categorical scoring chart for evaluating proof-writing skill in terms of the reasoning process

The pre-service teachers' proofs were first examined using the categorical scoring chart to identify the effect of the learning environment, based on the ISMAT Model, on their proof-writing skills. Examinations were conducted to evaluate inter-researcher agreement on coding reliability proof related to the test. A random sample of 30% was selected from the papers of both groups for these examinations. Before the evaluations, the categorical scoring rubric was presented to the other researcher, accompanied by clarifications and examples of indicators. The examinations indicated an 83% concordance in coding between the researchers. Follow-up discussions addressed coding inconsistencies, leading to necessary adjustments. An example of data analysis for PWPRE and PWPOST, as shown in the chart below (see Table 4), is provided.

The scoring for each question on the PAPRE and PAPOST was recorded in an Excel file and transferred to Winsteps for analysis. The points given to the proofs of the problems in the tests through this program were converted into linear points through Rasch analysis. These linear points were the pre-service teachers' achievement points. Rasch analysis was used to overcome the problems likely to result from the fact that the differences between the categories on the scoring chart were not equal. Statistical analyses were made with the linear points obtained through the Rasch analysis. The Mann-Whitney U test was used to determine whether there was a statistically significant difference in reasoning achievement between the experimental and control groups before the experiment. This test was chosen due to the independence of groups and the non-normal distribution of achievement points. Covariance analysis was conducted to assess the significance of the difference in reasoning achievements between the experimental and control groups after the experiment, and to determine if this difference was due to the experimental conditions.

Test	Proof for PWT	Data Analysis
PWPRE		By stating the congruence of the triangle, it was written as $[BFD] \cong [AFD]$. A similar notation was preferred when writing the congruence of the EGC and ELC triangles. Instead of writing |AE| = |DE|, it was stated as [AE] = [DE]. For the other side of equality, it was also stated with this symbol. In the proof, there are only deficiencies in the mathematical symbols. Therefore, this proof belongs to RP3.

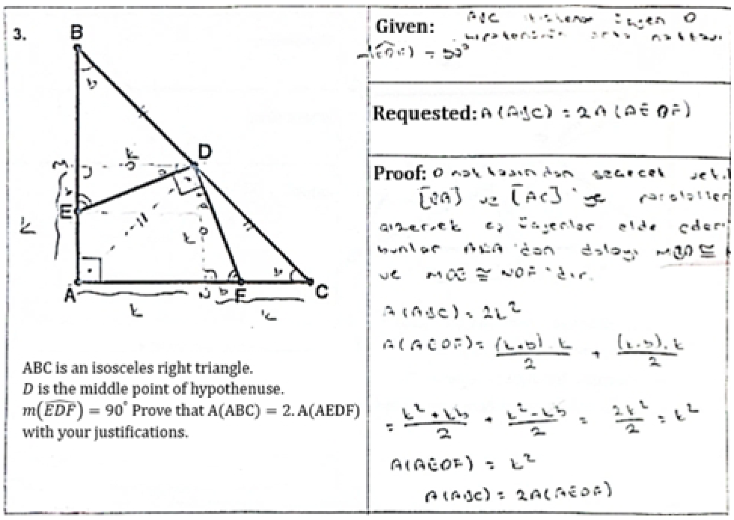
Test	Proof for PWT	Data Analysis
PWPOST		In the proof, the chain of inference is written appropriately. All the justifications between inferences were stated. Besides, when writing proof steps, mathematical language is taken into consideration. Therefore, this proof is in RP4.

Table 4: Example of data analysis for PWPRE and PWPOST

RESULTS

Results Concerning the Pre-service Teachers' Proof-Writing Skills before the Experiment

The pre-service teachers' proofs in the pre-test were evaluated, and then each of their proof-writing skills was determined. Table 5 presents the frequency and percentage distribution obtained from evaluating the proofs in PWPRE.

As Table 5 indicates, 39% of the experimental group proofs and 51.47% of the control group proofs were categorized as RP0 before the experiment, denoting a predominance of proof in this category among pre-service teachers. A significant number of these proofs merely restated the problem without generating inferences. Instances also occurred where pre-service teachers either omitted responses or included irrelevant statements that did not contribute to the proofs. Furthermore, 31% of the experimental and 28.8% of the control group proofs fell into RP1, which succeeded RP0. The majority of RP1 proofs in both groups (21.6% experimental, 19.35% control) included at least one correct independent inference, indicating a transition from RP0 to RP1, with a focus on hypothesis and conclusion details. Some instances contained inferences based on cases without sufficient justification, although at least one inference from the conclusion was made, albeit less frequently than correct independent inferences. Inferences from the conclusion were limited in occurrence across all reasoning process categories in both groups. Figure 4 illustrates a proof that corresponds to RP1, which was the most prevalent case in both groups before the experiment, following the identification of details on the hypothesis and conclusion.

Categories	Experimental Group		Control Group	
	f	%	*f*	%
RP0	150	39.00	173	51.47
RP1	119	31.00	97	28.88
RP2	70	18.30	45	13.39
RP3	27	7.00	16	4.77
RP4	18	4.70	5	1.49

Table 5: Frequency and Percentage Distribution Before the Experiment

The pre-service teacher illustrated the measure of arc AB because angle ACB is inscribed in it. However, she incorrectly inferred that line segment TC was a tangent and that the bisector of angle BTA passed through the circle's center, despite only one tangent being drawn from point T. Consequently, she believed that ray TR bisected arcs AB and AC equally, leading her to assume she had established the specified mathematical relationship. Thus, she presented a proof containing both correct and incorrect inferences, including at least one valid conclusion. The interview revealed her familiarity with the mathematical statements employed in her proof and the requisite actions, yet she struggled to articulate them effectively.

Some proofs by pre-service teachers fell into RP2. The control group produced fewer proofs in this category than the experimental group. While they made necessary inferences, they occasionally lacked formal justification or provided incorrect justifications. A similar pattern was observed in cases involving inferences from special cases. Additionally, in RP2 proofs, some pre-service teachers failed to make mutually supportive inferences, though this was less common across both groups compared to other RP2 cases. Before the experiment, pre-service teachers produced proofs in RP2 mainly when they could draw a conclusion. Therefore, when able to make the necessary inferences, they provided proofs that contained reasoning gaps. Figure 5 illustrates one proof corresponding to RP2.

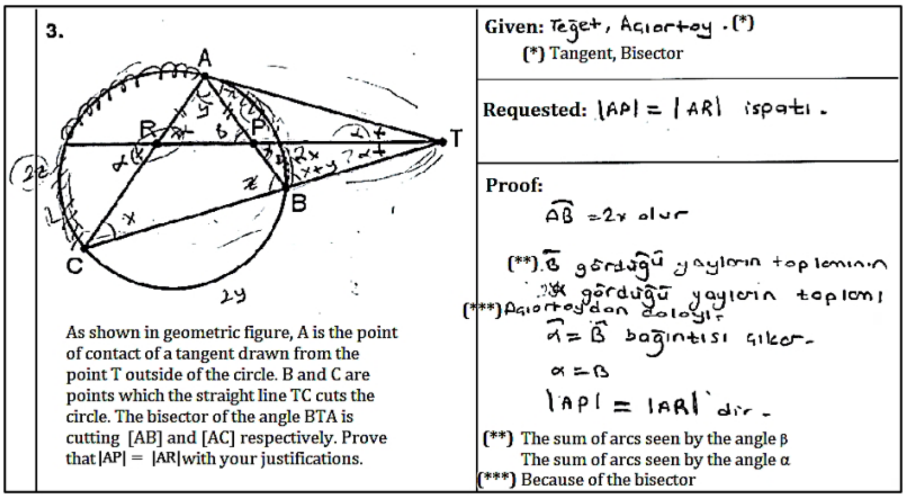


Figure 4: Example of a proof corresponding to RP1

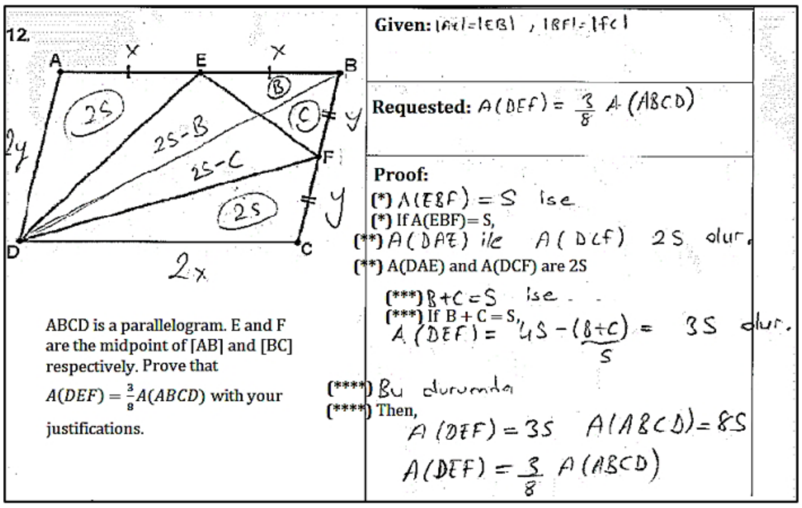


Figure 5: Example of a proof corresponding to RP2

As illustrated in Figure 5, the pre-service teacher defined the triangle EBF's area as S and deduced the equality of areas for triangles DAE and DEB; she calculated triangle DEF's area as 4S - (B + C) based on her notation; she concluded the area of triangle DEF is 3S; and the area of parallelogram ABCD is 8S. Despite making all the necessary inferences for the mathematical proof, she failed to provide justifications for these deductions. The interview revealed her awareness of her reasoning processes; she attributed her lack of justification to her prior experience of solving problems rapidly in mathematics courses.

The experimental group of pre-service teachers demonstrated a higher frequency of proofs in RP3 (7%) than the control group (4.77%). Justifications for selected proof-writing steps were more common than complete justifications for all steps. Instances of substantial step justifications with errors in terminology occurred, though these were less frequent. In RP4, where all inferences required for proof-writing were justified, the experimental group had a rate of 4.7%, compared to 1.49% in the control group. Nevertheless, the overall rate for this category was lower than that of other categories. Higher reasoning process categories (RP2, RP3, RP4) were observed less frequently in both groups than lower-level categories. This suggests that pre-service teachers struggled to identify all necessary proof-writing steps. It also reflects a deficiency in skills for justifying proof-writing steps or a perceived lack of necessity for such justifications. The infrequency of complete inferences and justifications indicates inadequate preparation for proof-writing before the experiment.

The Mann-Whitney U test, a non-parametric measure, was applied to the pre-test data to determine whether there was a statistically significant difference between the experimental and control groups in terms of reasoning achievement before the experiment. Table 6 presents the results of this test.

	Group	*N*	Mean Rank	Rank Sum	*U*	*p*
Pre-Test	Experimental	32	38.03	1217	207	0.000
	Control	28	21.89	613		

Table 6: Result of the Mann-Whitney U Test

Table 6 reveals that, prior to the experiment, there was a significant difference between the experimental group and the control group in terms of achievement in reasoning, favoring the former ($U = 207$, $p < 0.05$).

Results Concerning the Pre-service Teachers' Proof-Writing Skills After the Experiment

The pre-service teachers' proofs in the post-test were evaluated, and then each of their proof-writing skills was determined. Table 7 presents the frequency and percentage distribution obtained from evaluating the proofs in the PWPOST.

After the experiment, the proportions in RP0 were 14.06% in the experimental group and 43.45% in the control group. This suggests that the control group predominantly provided proofs in RP0. Conversely, the experimental group exhibited a significant reduction in RP0 proofs relative to the baseline, highlighting a notable disparity between the groups. The proofs in RP1 were 31.78% in the experimental group and 36.31% in the control group. Despite being less frequent in the experimental group, RP1 had the highest proportion among its categories. Most RP1 proofs in both groups involved "*making at least one correct independent inference*."

The RP2 category was more prevalent in the experimental group (27.08%) compared to the control group (12.2%). Despite its higher occurrence in the experimental group, the case of "*failing to attain the result by making inferences that support one another*" was common in both groups. This indicates that, post-experiment, pre-service teachers increasingly focused on the interconnections of inferences during proof-writing. Figure 6 illustrates a proof example related to RP2.

Categories	Experimental Group		Control Group	
	f	%	*f*	%
RP0	54	14.06	146	43.45
RP1	122	31.78	122	36.31
RP2	104	27.08	41	12.2
RP3	65	16.92	22	6.55
RP4	39	10.16	5	1.49

Table 7: Frequency and percentage distribution after the experiment

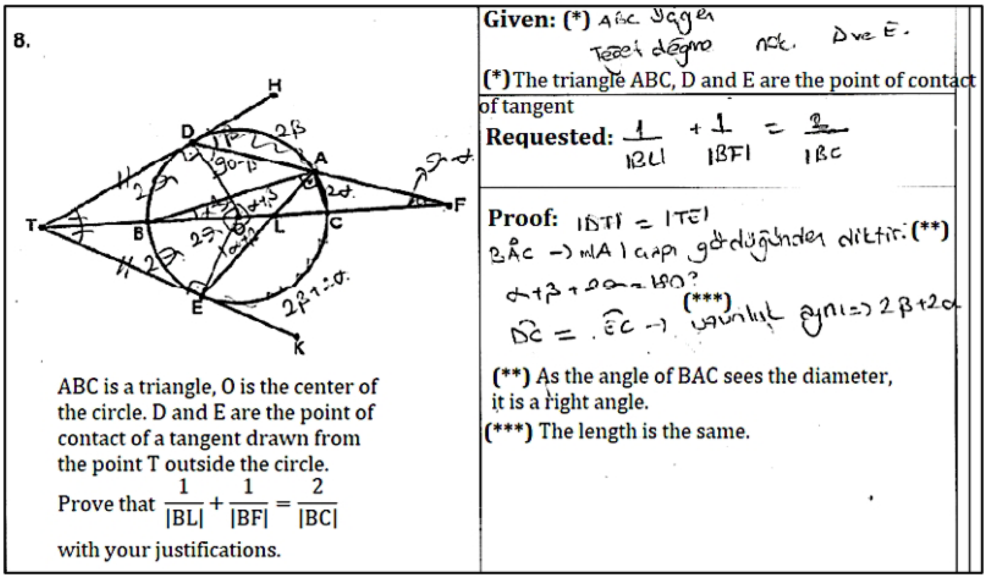


Figure 6: Example of a proof corresponding to RP2

As illustrated in Figure 6, the pre-service teacher inferred |DT| = |TE| from a tangent at point T. She established angles TDO and TEO as 90 degrees, based on the perpendicularity of the radius to the tangent. She deduced that angles DTO and ETO are equal because the line segment TF intersects the center of the circle. Consequently, she claimed that arcs DB and BE, as well as arcs DC and EC, were equal in measure. Additionally, she posited that angle BAC measures 90 degrees, justifying her assertion by its opposition to the diameter of the circle. Thus, she constructed a series of interconnected inferences. Nevertheless, her proof remained incomplete due to her omission of necessary concluding inferences.

The RP3 category proofs were more frequent in the experimental group (16.92%) than the control group (6.55%). Control group students exhibited a significantly lower percentage of RP3 proofs relative to their experimental counterparts. In both cohorts, the predominant proofs involved justifying certain steps in the proving process, with greater frequency in the experimental group. Limited instances of proofs that justified substantial steps but included errors in terminology were exclusive to the experimental group. This suggests that post-experiment, pre-service teachers, particularly in the experimental group, recognized the necessity for justifying their inferences related to proofs. Figure 7 illustrates an example of a proof relevant to RP3.

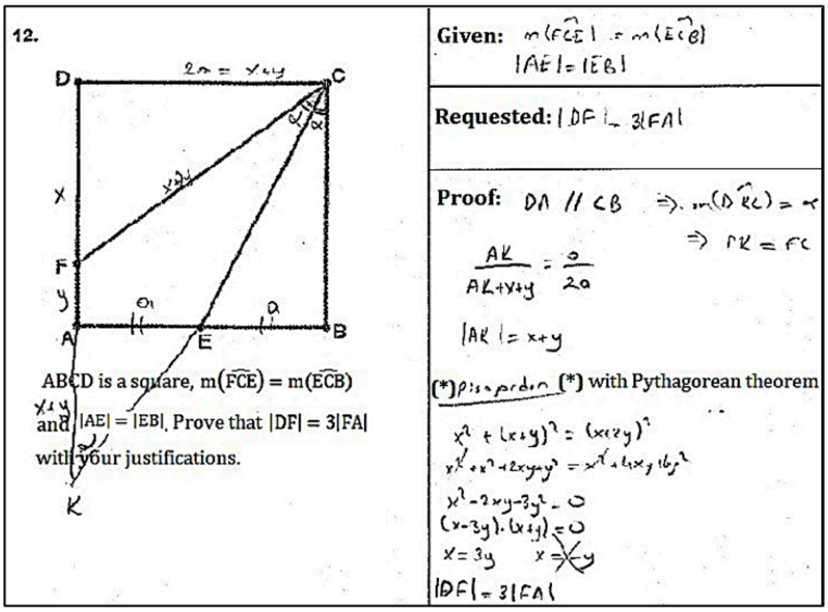


Figure 7: Example of a proof corresponding to RP3

As illustrated in Figure 7, the pre-service teacher articulated that $m\left(\widehat{DKC}\right)=\alpha$ by asserting the parallelism of line segments DA and CB, attributing this to quadrilateral ABCD being a square. He failed to justify the operation "$\frac{AK}{AK+x+y}=\frac{a}{2a}$", which he executed based on his personal interpretation of letters and supplementary drawings. He indicated that the operation he initiated as "$x^2+\left(x+y\right)^2=\left(x+2y\right)^2$" was predicated on the letters assigned to the side lengths in accordance with the Pythagorean Theorem. Thus, while he provided rationales for certain inferences made during the proof, he omitted justifications for others.

The experimental group showed a proof rate of 10.16% for RP4, while the control group exhibited a rate of 1.49%. Nevertheless, both groups had lower rates in this category compared to others. Additionally, the control group showed no change in RP4 proof rates pre- and post-experiment. In contrast, the experimental group experienced an increase in RP4 proof rates. An illustration of an RP4 category proof is presented in Figure 8.

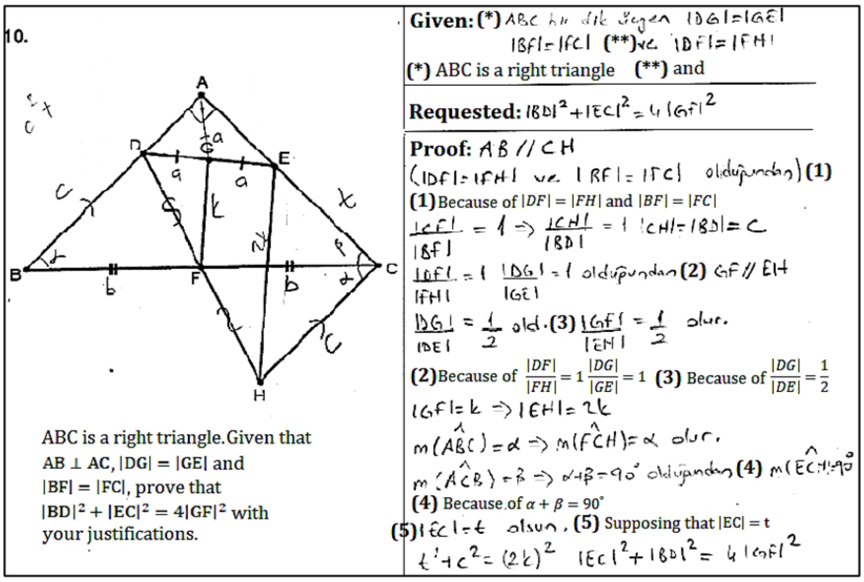


Figure 8: Example of a proof corresponding to RP4

As illustrated in Figure 8, the pre-service teacher asserted that the ratio of the triangles DFB and HFC was equivalent, leading to the conclusion that line segments AB and CH are parallel. She claimed that the lengths of segments BD and CH were identical and provided sound reasoning for this conclusion. Her assertion of the parallelism of sides GF and EH, based on the side length ratio, constituted a valid step in the proof. She established angle measures accordingly and deduced that angle ECH is 90 degrees, offering a well-founded justification for this inference. Her utilization of the Pythagorean Theorem, predicated on this angle measurement, represented a legitimate and conclusive step in the proof. Consequently, she delivered a comprehensive proof in RP4 by articulating each necessary inference and thoroughly substantiating each assertion.

According to the categories of the reasoning process, the comparison of the proof rates for the experimental and control groups before and after the experiment is shown in Figure 9.

As Figure 9 demonstrates, the experimental group primarily produced proofs in all reasoning categories except RP0 prior to the experiment. Post-experiment, the control group favored proofs in RP0 and RP1, while the experimental group dominated in RP2, RP3, and RP4. The control group showed a higher prevalence of non-proof statements compared to the experimental group. In contrast, the experimental group presented a larger number of statements with proof quality, despite some shortcomings in mathematical language or justification.

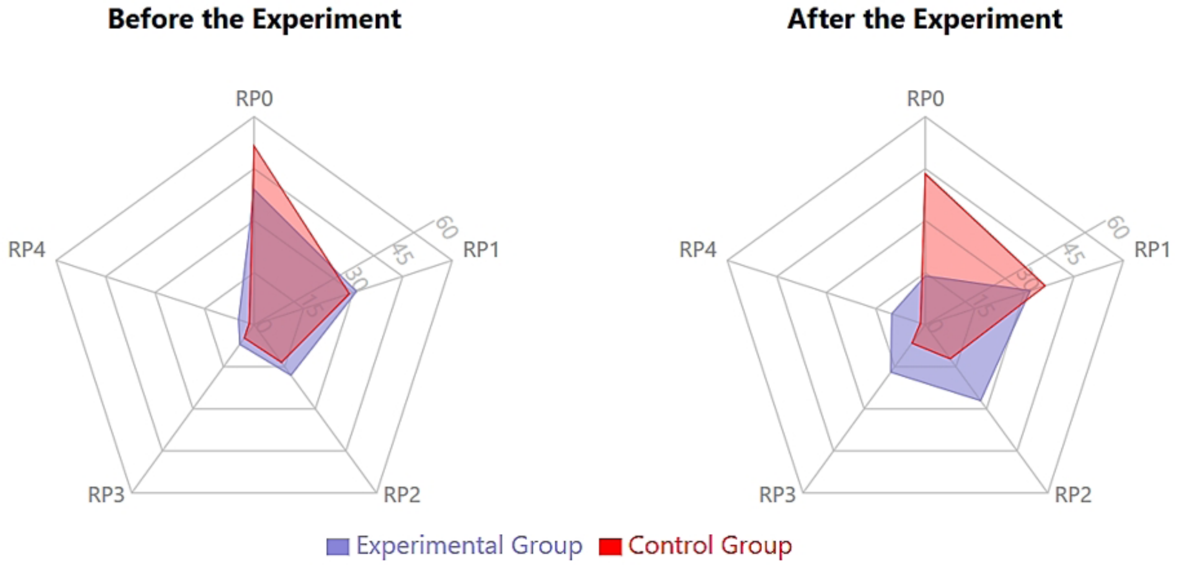


Figure 9: Radar graph for comparison of the groups' proof rates before and after the experiment

An analysis of covariance (ANCOVA) was carried out using the pre-test scores as the "covariate" to determine whether there was a significant difference between the groups' post-test scores concerning their achievement in reasoning, and if so, whether such a difference really resulted from the experimental conditions. Table 8 presents the descriptive statistics for the mean post-test scores, along with the reasoning behind the calculations, as well as the adjusted mean post-test scores. Table 9 presents the results of the ANCOVA, demonstrating whether the difference between the adjusted mean post-test scores of the groups is statistically significant.

Group	n	Post-Test Score		Adjusted Post-Test Score	
		$\bar{X}$	SD	$\bar{X}_a$	SH
Experimental Group	32	0.35	0.47	-0.129	0.152
Control Group	28	-0.84	1.35	-0.652	0.163
Total	60	-0.37	1.07		

$\bar{X}_a$: Adjusted Mean Post-Test Score

Table 9: ANCOVA results concerning the post-test scores concerning achievement in reasoning in the proving process

Source of Variance	Sum of Squares	Sd	Mean Square	F	Level of Significance	Effect Size (eta-square)
Pre-test	16.277	1	16.277	23.144	0.000	0.289
Method	3.681	1	3.681	5.234	0.026	0.084
Error	40.089	57	0.677			
Total	76.111	60				

Table 8: Descriptive statistics of the post-test scores concerning the experimental and control group students' achievement in reasoning in the proving process

According to the results of the ANCOVA, as shown in Table 9, there was a statistically significant difference between the post-test scores of the experimental group and those of the control group when the two groups' pre-test scores were controlled ($F_{(1, 57)} = 5.234$, $p < 0.05$). In other words, the improvement in the pre-service teachers' achievement in reasoning was associated with the activities carried out in the learning environment based on the ISMAT Model. Accordingly, the lessons on proof conducted in the learning environment designed for the experimental group contributed to the improvement in the pre-service teachers' achievement in reasoning.

DISCUSSION AND CONCLUSION

Progress in pre-service teachers' reasoning was observed within the ISMAT Model. The experiment revealed a decline in proofs for the lower reasoning categories (RP0, RP1), while proofs for the higher reasoning categories (RP2, RP3, RP4) increased. The experimental group experienced a decrease from 39% to 14.06% in the RP0 category, representing a reduction of more than 50%. Conversely, the control group experienced a minor decrease from 51.47% to 43.45% in the same category. The experimental group exhibited significant changes compared to the control group. This may be attributed to the researcher's emphasis on distinguishing between given conditions and required proofs. Consequently, pre-service teachers faced fewer challenges in initiating mathematical proofs. The rate of cases corresponding to RP0 decreased from 39% to 14.06%, indicating an increase in proof attempts. The success in initiating proofs likely bolstered their confidence and reduced their inclination to leave questions unanswered. It can be posited that heightened motivation and positive emotions facilitated deeper study and enhanced learning efforts (Heinze & Reiss, 2009). Although some proofs remained incomplete, increased self-confidence fostered a reasoning mindset that initiated the process of proving. Thus, post-experiment, pre-service teachers employed various reasoning methods to initiate proofs. Moore (1994) suggested that a lack of conceptual understanding can hinder the initiation and execution of proof. In alignment with this, the current study indicates that the pre-service teachers' conceptual understanding improved through the use of the ISMAT Model.

The proportion of proofs in RP2 increased from 18.3% to 27.08% post-experiment. This indicates that the experimental group improved in identifying all the necessary steps required to complete the proof. Consequently, a transformation occurred in the integration of supporting inferences into a cohesive presentation. This shift likely enhanced the pre-service teachers' reasoning skills, fostering a comprehensive perspective. This finding aligns with previous research indicating that reasoning improvement activities positively influence learners who initially exhibited inadequate proof explanations (Driscoll, 1987; Lee, 1999; Mata-Pereira & da Ponte, 2017; Moore, 1994; Schoenfeld, 1985).

The occurrence rate of proofs in RP3 increased from 7% to 16.92% post-experiment. This denotes a more than two-fold enhancement in proof occurrence. It suggests that pre-service teachers recognized the significance of justifying their inferences and integrating various inferences cohesively. Additionally, it indicates a tendency to scrutinize the basis of their expressed inferences. Pre-service teachers exhibited increased awareness of mathematical expression utilization. Such advancements likely stemmed from discussions surrounding the mathematical expressions and justifications presented during the proving of relationships. These improvements may have been promoted by discussions in the experimental group regarding the suitability of expression, grounds for inferences, and the relevance of justification. Lee (1999) articulated that prompting students to elucidate their reasoning enhances their proof-writing skills. Nonetheless, the proof count in RP3 fell short of expectations, possibly due to some group members merely observing and recording rather than actively engaging. Students who did not critically engage with the proof steps may not have achieved the same advancements in individual proof-writing assessments.

The occurrence rate of proofs in RP4 increased from 4.7% to 10.16%, indicating a significant rise. This suggests that pre-service teachers enhanced their awareness of the necessary inferences for proofs and justifications. Their improved expression of inferences and justifications reflects a solid grasp of reasoning processes. This progress likely stemmed from the experimental learning environment, which facilitated discussions on the correctness and relevance of inferences and justifications. However, these advancements were insufficient, potentially due to a lack of focus on improving spatial skills, such as drawing. Consequently, the anticipated increase in proofs may not have materialized, as additional drawing questions would require deeper, multi-dimensional reasoning. Additionally, some pre-service teachers' increased efforts in group work to identify and justify proof-writing steps may have influenced these outcomes.

The frequency of higher reasoning proofs (RP2, RP3, RP4) among pre-service teachers increased post-experiment. This indicates that the ISMAT Model learning environment enhanced their reasoning in the proving process. Control group proof data from before and after the experiment further corroborates this finding. Notably, proofs in RP0 were most prevalent after the experiment, followed by RP1. While a minor rise in RP3 proofs was observed, RP2 experienced a decline, and RP4 remained stable. This suggests that proof teaching for the control group had minimal impact on their reasoning. Consequently, the experimental group's proof teaching effectively improved their reasoning skills. Previous research has similarly highlighted that various educational strategies fostered student reasoning development (Erdem, 2015; Francisco & Maher, 2005; Generazzo, 2011; Hiebert & Grouws, 2007; Hsu, 2010; Lee, 1999; Martin & McCrone, 2009; Pulley, 2010; Reiss, Hellmich & Reiss, 2002).

Proof evaluation activities enhance students' reasoning, aligning with Pulley's (2010) findings that such activities dispel misconceptions and strengthen understanding of mathematical proofs. These activities are pivotal for pre-service teachers, facilitating the analysis of peer proofs, comparison with their own inferences, and fostering awareness of diverse proof-writing techniques.

We assert that the pre-service teachers' presentations may have transformed their reasoning through exposure to diverse proof-writing techniques. This assertion is corroborated by various studies (e.g., Generazzo, 2011; Pulley, 2010; Weber et al., 2008), which indicate the beneficial impacts of class discussions on the learning process. Generazzo (2011) posits that designated time for class discussions can enhance both reasoning and proof-writing skills, fostering positive shifts in students' perspectives on proof-writing. In our study, interactive group work, peer sharing of insights, and the opportunity to recognize and rectify mistakes contributed to advancements in reasoning during the proving process. Furthermore, numerous researchers emphasize the significance of group work in proof teaching, noting that interactive environments facilitate reasoning enhancement (Generazzo, 2011; Haralambos, 2000; Lee, 1999; Moreno, 2003; Pulley, 2010; Tinto, 1990). Yankelewitz, Mueller and Maher (2010) similarly highlight that settings that promote peer interaction and the expression of mathematical ideas are optimal for developing mathematical reasoning. In this context, drawing from Vygotsky's (1978) perspective, it can be concluded that reasoning improvement occurs in socially interactive environments, as individuals are influenced by the reasoning of their peers (Maher & Davis, 1995).

The structural alignment of the ISMAT Model with the methodological procedures employed by professional mathematicians provides pre-service mathematics teachers with a systematic framework for developing proofs, which contributes significantly to both pedagogical efficiency and professional accountability. Results supported by Buchbinder and McCrone (2023) demonstrate that structured module-based approaches enhance pre-service teachers' content and pedagogical knowledge regarding the role of examples and quantifiers in proofs. The model optimizes time-to-competency ratios while enhancing the quality of evidence-based reasoning relative to initial knowledge levels, supported by Al-Sa'ad and Alzoebi (2024), who confirmed that systematic training programs based on NCTM standards produce statistically significant gains in teachers' pedagogical knowledge across multiple dimensions. Furthermore, ISMAT's systematic approach cultivates metacognitive awareness among pre-service teachers, aligning with the pedagogical-metacognition model proposed by Kohen and Kramarski (2018), and strengthening their ability to identify and correct errors within their own learning processes. Evidence from Erdoğan and Kalkan (2024) reveals that metacognitive awareness explains 38% of the variance in critical thinking scores, supporting a transformative shift from passive knowledge acquisition to active knowledge construction. The socio-constructivist dimension, supported by Generazzo (2011) and Pulley (2010) and enhanced by collaborative evidence-based reasoning processes emphasized by Csanádi et al. (2021), demonstrates that the model extends beyond individual cognitive development to encompass collaborative mathematical discourse and peer-mediated learning experiences, positioning the ISMAT Model as a theoretically grounded, empirically supported, and pedagogically innovative approach in mathematics teacher education.

Several limitations must be recognized when analyzing these results. Primarily, the research was conducted within a single institution, which may limit the applicability of the findings to alternative educational settings characterized by diverse student demographics, institutional cultures, or resource availability. Secondly, the hierarchical organization of the data, with students organized within classrooms, may have introduced interdependencies that could potentially influence the statistical interpretations, as conventional analyses presuppose the independence of observations. Thirdly, the evaluation was confined to assessments conducted immediately following instruction without any longitudinal follow-up, thereby rendering uncertain whether the noted enhancements in proof-writing skills are sustained over time or merely indicative of transient improvements. Lastly, the experimental group's exposure to innovative teaching methods and dynamic geometry software may have resulted in enhanced performance due to Hawthorne or novelty effects rather than the inherent effectiveness of the ISMAT Model itself. The increased attention and motivation from participating in a novel educational approach could have confounded the true impact of the intervention. Future studies should address these limitations through multi-institutional studies, appropriate statistical modeling for nested data, longitudinal follow-up assessments, and careful control for attention effects to provide more robust evidence of the ISMAT Model's effectiveness in developing proof-writing skills.

Based on the results of this research, it is recommended that proof teaching be grounded in real-life mathematical activities. Moreover, instructional models should facilitate student involvement in the proving process. Additionally, since proof teaching is crucial at all educational levels, the ISMAT Model could be adapted accordingly. Consequently, further research is warranted to evaluate the effectiveness of the ISMAT Model across various educational stages.

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