STUDENTS' MATHEMATICS KNOWLEDGE – THEORY AND EXERCISES

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Highlights

- *Knowledge tests – relationship between the ability to formulate definitions and to solve exercises*
- *Most common errors in formulations of the selected definitions*

Abstract

The article analyses the test results evaluating the knowledge of students of basic mathematics courses at the University of Economics in Prague and at the University of Finance and Administration in Prague. The relationships between the study of the theory, the ability to formulate definitions and to solve exercises are analysed based on the results in two groups of students of the University of Economics. For this purpose the statistical evaluation utilizing the log-linear models is used. The success rate in formulating definitions in two groups of students of both universities is compared using hypothesis testing. The most common errors in the theoretical parts of the tests are presented.

Keywords

Definition, exercise, log-linear model, mathematics, test, theory

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Introduction

The article is an extension of the paper presented at the 12th International Conference on Efficiency and Responsibility in Education (ERIE 2015) (Ulrychova, 2015).

The article deals with the teaching of mathematics (the specific parts of linear algebra, in particular) at two universities of economic studies in Prague – at the University of Economics (a public school) and at the University of Finance and Administration (a private school). It focuses especially on the theoretical part – it is assessed whether the knowledge of theory affects the ability to solve exercises positively. Moreover, the success in the formulation of definitions is compared for students of both universities and most common errors in the theoretical part are presented. These results lead to ponder to what extent it is necessary to emphasize theory and to demand the exact wording of the definitions in mathematics education at universities of economic studies.

Students' performance at universities can depend on their high school mathematics knowledge (cf. (Kučera and Svatošová and Pelikán, 2015)). Some students, who were not successful in their mathematics study at high school, suffer from mathematics anxiety (Ashcraft and Moore, 2009) at universities. Many students dislike mathematics based on their experience at high schools.

Students of universities of economic studies are usually not proficient enough in basics of high school mathematics and they are not used to formulate mathematical definitions and theorems. They were not led to the logical thinking; they were taught to math skills without real understanding of the matters. It is difficult to change the students' approach to the study of mathematics at university, particularly when a small number of lessons is available.

Students prefer comprehensible way of teaching, not very focused on the theory – such a way that does not stress them by its complicated form. This also corresponds to experience with other European universities – see e.g. (Widenská, 2015). For example, Oldknow (2009) and Widenská (2014) consider the including of information and communication technology as a way of making the study of mathematics more attractive.

The mathematics curriculum at the University of Finance and Administration is taught with a stronger emphasis on the understanding against a memorization only. Therefore, the curriculum is interpreted as simply as possible in mathematics lessons and – compared to the University of Economics – students are allowed to formulate the theory less formal during the oral exam, e.g. by words instead of a formal mathematical notation. In this article, we compare the results in groups of students of both universities in order to find out if this way of interpreting of mathematics curriculum brings better results. However, the results can be also affected by different levels of students' skills regarding to different characters of the universities (a public and a private school).

The basic mathematics course at the University of Economics in Prague is taught for a period of one semester within the range two lessons of lectures and two lessons of seminars per week. It includes the basics of mathematical analysis and linear algebra in the range corresponding with textbooks such as (Batíková, 2009) or (Klůfa and Kaspříková, 2013).

Except for the final exam consisting of a written test and an oral part at the end of the semester, students of mathematics course are tested using a written test in the middle of the semester. The result of this test is counted towards the result of the final exam – more in (Otavová and Sýkorová, 2014). Until the academic year 2009/2010, these mid-term tests consisted of exercises only. In summer semester 2009/2010, the author of this article included (as a part of the research for her doctoral thesis) an extra task to formulate given definitions into the mid-term test.

The basic mathematics course at the University of Finance and Administration is taught for a period of two semesters within the range two lessons of lectures and one lesson of seminars per week. It includes curriculum similar to the curriculum at the University of Economics – see the textbook (Budinský and Havlíček, 2005).

In each of the two semesters, the course is completed with a credit and an exam. To award the credit, students are required to pass a written test. The credit is a prerequisite for taking the exam. The exam consists of a written part and a verbal part; prerequisite for taking the verbal part of the exam is to pass the written part. The both credit and exam written tests consist of exercises only. In the academic year 2015/2016, the author of this article included into the credit test at the University of Finance and Administration the same extra task as at the University of Economics in the academic year 2009/2010.

The primary aim of this article is to evaluate the effect of the study of the theory for the results of the tests and to assess the relationships between the study of the theory, the ability to formulate the definitions and the ability to solve exercises. Another aim is to compare results at two similarly oriented universities. Although the research was focused primarily on tests from the area of linear algebra and on students of the University of Economics (see also (Kaspříková, 2012)) and of the University of Finance and Administration, the results could be transferable to other fields of mathematics and other similarly oriented universities (see e.g. (Milková, 2011), (Brožová and Rydval, 2014)).

Materials and Methods

In the academic year 2009/2010, three teachers of the Department of Mathematics at the University of Economics in Prague administered mid-term tests with the theoretical part to 300 students of different study groups and fields of study. Students were required to formulate following five definitions: linear combination of vectors, linear dependence of vectors, rank of a matrix, invertible matrix, matrix inversion. They had not expected this theoretical part in the test and it can be assumed that they had not prepared for that. Students were informed that the results of this part would not be taken into account for the official rating of the tests and they were asked to try to formulate the definitions in the best way they can.

The theoretical part in all the tests was evaluated by the author of this article. The correctness of formulations and the most common errors were determined (in detail in (Ulrychová, 2013)). The evaluation of the accuracy of formulations was very moderate – for example, the non-generic definition of linear combination of vectors expressed just for two vectors was accepted as correct. In addition, in the case of tests comprising an exercise related directly to one of the five given terms, the relationship between the correctness of the solution of the exercise and the correctness of the formulation of relevant definition was examined.

In the academic year 2010/2011, the head of Department of Mathematics made it mandatory to include a task to formulate one definition or theorem (not necessarily from linear algebra field) in all mid-term tests. Students were informed in advance about this fact and the results of this part were counted towards the official rating of the tests. In that year, the author of this article took the exceptional opportunity to compare the results of the tests in the group of students who did not expect the theoretical part in the test (the group *A*) with the results of tests in the group of students who did expect this part (the group *B*).

In the academic year 2015/2016, the author of this article repeated the experiment from year 2009/2010 (the group *A*) under the same conditions by administering credit tests at the University of Finance and Administration (the group *C*).

Results of all students (of chosen teachers), who passed mid – term tests (credit tests, respectively) in regular terms, are included in the experiment in all three cases (the groups *A*, *B*, *C*). The results of students in these groups are compared. Attention is focused on the theoretical part; the ability to formulate the definitions is assessed and the most common errors are described.

The character of the groups *A* and *B* allows us to assess the effect of study of the theory on results in both theoretical and practical parts of the test. For this purpose, the relationships between the study of the theory, the ability to formulate definitions and to solve exercises are analysed using log-linear models (Agresti, 2002) for the groups *A* and *C*.

Let´s denote T = the student did/did not expect the theory in test, E = the exercise was/was not correct and D = the definition was correct/incorrect.

The level of dependence in each of pairs „the student can/cannot formulate the definition – the student can/cannot solve the exercise" (pair DE); „the student can/cannot formulate the definition – the student did/did not expect the theory in test" (pair DT); „the student can/cannot solve the exercise – the student did/did not expect the theory" (pair ET) is examined.

The log-linear hierarchical models (saturated, homogeneous association, conditional independence, joint independence) and function "glm" (generalized linear models) in R-software are used to determine the best model in each category. The statistical tests of their feasibility were performed using the standard statistical testing of submodel (the deviance test) (see (Agresti, 2002)).

The saturated model corresponds to reality (obtained data). In the homogeneous association model all three pairs DE, DT, ET were retained (denoted DE.DT.ET). In the conditional independence model (a reduced model of the homogeneous association model) one of the pairs DE, DT, ET was always omitted. Among these reduced models, the one which best coincided with the reality (and with the homogeneous association model) was chosen. The pair, whose omission leads to the least breach of the accordance with reality, shows the weakest relationship between its members (compared to the other two pairs). In the joint independence model (a reduced model of the conditional independence model) another pair was omitted and again the model conforming the best with the reality was chosen. In the pair, which remained as the last, the relationship between its members is the strongest compared to the other two pairs.

The character of the groups *A* and *C* allows us to compare the success in formulation of definitions of students of two similarly oriented universities. The hypothesis test of equality of relative frequencies of alternative distribution is used for comparison of the results in groups *A* and *C* – see e.g. (Bílková, Budinský and Voháňka, 2009).

In two independent random samples (the groups A and C) of large sizes n_1 and n_2 the null hypothesis H_0: $\pi_1 = \pi_2$ is tested at the 5% significance level against the alternative hypothesis H_1: $\pi_1 \neq \pi_2$, $\pi_1 < \pi_2$ and $\pi_1 > \pi_2$, respectively. The test criterion

$$U = \frac{P_1 - P_2}{\sqrt{\overline{P}\left(1-\overline{P}\right)}} \cdot \sqrt{\frac{n_1 \cdot n_2}{n_1 + n_2}}, \qquad (1)$$

where

$$P_1 = \frac{M_1}{n_1}, P_2 = \frac{M_2}{n_2} \qquad (2)$$

(relative frequencies) and

$$\overline{P} = \frac{M_1 + M_2}{n_1 + n_2}, \qquad (3)$$

is used.

The corresponding critical ranges are $W\alpha = \{u;\ |u| \geq u_{1-\alpha/2}\}$ for the alternative hypothesis H_1: $\pi_1 \neq \pi_2$, $W\alpha = \{u;\ u \geq u_{1-\alpha}\}$ for H_1: $\pi_1 > \pi_2$ and $W\alpha = \{u;\ u \leq - u_{1-\alpha}\}$ for $\pi_1 > \pi_2$.

For $\alpha = 0.05$, the critical values are $u_{1-\alpha/2} = u_{0.975} = 1.960$ and $u_{1-\alpha} = u_{0.95} = 1.645$.

Results

Evaluation of Tests in the Group A (The University of Economics in Prague)

The group A is the group of 300 students who did not expect the theoretical part in the test. Table 1 shows the number of correct answers in group A.

	number	%
linear combination	93	31%
linear dependence	89	29.67%
matrix rank	186	62%
invertible matrix	205	68.33%
matrix inversion	95	31.67%

Tab. 1: Success rate in formulating particular terms (the group *A*)

As we can see in Table 1, in three cases (linear combination of vectors, linear dependence of vectors, matrix inversion) the success rate was about 30%, in the case of rank of a matrix and invertible matrix the success rate was more than double (62% and 68% respectively). Interestingly, the terms rank of a matrix and invertible matrix are built on the term linear dependence and hence on the term linear combination, formulation of which was much less successful.

In addition, the relationship between knowledge of the term rank of a matrix and invertible matrix, respectively, and the terms on which these terms are built (linear dependence and linear combination) was examined. As we can see in Table 2, 186 students (out of 300 total) defined correctly the term rank of a matrix but only 41 (i.e. 22%) of them defined correctly the remaining terms. Only 40 (i.e. 19.51%) out of 205 answers was correct in the case of the invertible matrix.

In the following tables, "yes/no" means "the definition (exercise respectively) is correct/incorrect".

MATRIX RANK yes **186**				INVERTIBLE MATRIX yes **205**			
Linear dependence yes **67**		Linear dependence no 119		Linear dependence yes **71**		Linear dependence no 134	
Lin. comb. yes **41**	Lin. comb. no 26	Lin. comb. yes 29	Lin. comb. no 90	Lin. comb. yes **40**	Lin. comb. no 31	Lin. comb. yes 33	Lin. comb. no 101

Tab. 2: Definitions (matrix rank, invertible matrix)

In addition, 230 out of 300 tests included an exercise related either to linear dependence (100 students) or to matrix inversion (130 students). In these cases the relationship between the correctness of the solution of the exercise and the correctness of the formulation of relevant definition was examined. Considering this relationship, only the correctness of the solution procedure of the exercise (not numerical errors) was taken into account. The evaluation is in Table 3. The differences in data in Table 1 and Table 3 are given by the fact that the data in Table 1 are related to the total number 300 students, whereas the data in Table 3 are related to the total number 100 (in the case of linear dependence) or 130 (in the case of matrix inversion).

Evaluation of Tests in the Group B (The University of Economics in Prague)

The group B is the group of 230 students who expected the theoretical part in the test.

In the academic year 2010/2011, the task to formulate one given definition or theorem (not necessarily from linear algebra field) was mandatory in every mid-term test. It was impossible to incorporate more than one definition into the test, therefore it was not possible to make a direct comparison to the year 2009/2010 in all aspects. In order to make it possible to compare to the year 2009/2010, the relationship "knowledge of definition – correctness of the solution procedure of the exercise", exercises of the same type and definitions related to them as in 2009/2010 were given. The tests intended for comparison with the year 2009/2010 were administered by the same three teachers as in 2009/2010 to the same number of students taking the test allowing to compare the relationship between the definition and the exercise (i.e. to 230 students). The total number of tested students in the group B (230 students) was smaller than in the group A (300 students) and only the definitions of the terms linear dependence of vectors (100 students) and matrix inversion (130 students) were asked in the group B. The assessment criteria were the same in A as in B.

The number of correct answers and the evaluation of the relationship "knowledge of definition – correctness of the solution procedure of the exercise" are summarized in Table 3.

Results in the Groups *A* and *B*

Table 3 shows summary results in both groups A and B.

Students in the group A did not expect the theoretical part in the test – we can assume that they probably had not studied the theory. Students in group B expected the theoretical part in the test – we can assume that they had studied the theory. We are interested in how this fact is reflected in the results of tests – the level of dependence in each of pairs „the student can/cannot formulate the definition – the student can/cannot solve the exercise"; „the student can/cannot formulate the definition – the student did/did not expect the theory in test"; „the student can/

cannot solve the exercise – the student did/did not expect the theory" is studied.

	LINEAR DEPENDENCE				MATRIX INVERSION			
	A (in total 100)		*B* (in total 100)		*A* (in total 130)		*B* (in total 130)	
	number	%	number	%	number	%	number	%
Definition								
yes	36	36	42	42	43	33.08	88	67.69
no	64	64	58	58	87	66.92	42	32.31
Exercise								
yes	90	90	88	88	97	74.62	93	71.54
no	10	10	12	12	33	25.38	37	28.46
Def. – Ex.								
yes – yes	35	35	40	40	39	30	66	50.77
yes – no	1	1	2	2	4	3.08	22	16.92
no – yes	55	55	48	48	58	44.62	27	20.77
no – no	9	9	10	10	29	22.31	15	11.54

Tab. 3: Results in *A* and *B*

The results of the statistical evaluation by the log-linear models (in accordance with the notation above) are presented in the following tables. The first one shows the predicted counts in each category, the second table presents the fit of the model following (Agresti, 2002) in each of the two cases – linear dependence and matrix inversion. In logistic regression, the residual sum of squares is usually replaced by the deviance G^2 (Agresti, 2002). For two nested models, the difference in deviances has an asymptotic chi-squared distribution with degrees of freedom equal to the difference in the degrees of freedom for the two models. G^2 statistics is used to test the null hypothesis that the model holds against the saturated model. The bigger the value of G^2 (unbounded), the more we tend to reject the null hypothesis (i. e. the tested model). The quantity df is the degree of freedom and p-value is the probabilistic level on that the null hypothesis is (or is not) denied. The right column (delta) shows the dissimilarity index.

1) Linear dependence

Table 4 shows the predicted counts in each category, Table 5 presents the goodness of fit test and dissimilarity index (delta) of the models.

LINEAR DEPENDENCE							
	Definition	Exercise	Theory	Number	DE.DT.ET	DE.DT	DE.T
1	yes	yes	no	35	35	35	37
2	yes	no	no	1	1	1	2
3	no	yes	no	55	55	54	52
4	no	no	no	9	9	10	9
5	yes	yes	yes	40	40	40	37
6	yes	no	yes	2	2	2	2
7	no	yes	yes	48	48	49	52
8	no	no	yes	10	10	9	9

Tab. 4: The best models – linear dependence

	G^2	df	p-value	delta
fitDET	0.0	0	1.00000000	0.000
fitDE.DT.ET	0.1	1	0.81071295	0.004
fitDE.DT	0.4	2	0.80054144	0.014
fitDE.T	1.2	3	0.75249357	0.035

Tab. 5: Goodness of fit of the linear dependence model

Table 5 confirms that none of the tested models is denied at the 5% significance level.

All models DE.DT.ET, DE.DT and DE.T fit the data sufficiently well. The model DE.DT omits the insignificant ET pair, meaning that given level of D, E and T are independent. Thus the weakest relationship is in the pair ET – in the group of students having (or not having) the definition correct, the correctness of exercise is independent on study of the theory; i.e. the ability to solve exercises is independent on study of theory. The model DE.T shows that the strongest relationship is between D and E; i.e. the relationship between correctness of the definition and the exercise.

The odds ratio of DE was calculated from the model DE.T: let Dy, Dn denote „definition yes“ (the definition was correct) and „definition no“ (the definition was incorrect), analogically for the exercise Ey and En. Then (Dy/Dn)/(Ey/En) = exp(1.529) = 4.614, meaning that increasing the ratio Ey/En, the ratio Dy/Dn increases about 4.6 times. In other words, if the ratio of students having the exercise correct to students not having the exercise correct increases, the ratio of students having the definition correct to students not having the definition correct increases 4.6 times.

2) Matrix inversion

Table 6 shows the predicted counts in each category; Table 7 shows the goodness of fit test and dissimilarity index (delta) of the models.

MATRIX INVERSION							
	Definition	Exercise	Theory	number	DE.DT.ET	DE.DT	DE.T
1	yes	yes	no	39	37	34	31
2	yes	no	no	4	6	9	12
3	no	yes	no	58	60	57	64
4	no	no	no	29	27	30	23
5	yes	yes	yes	66	68	71	64
6	yes	no	yes	22	20	17	24
7	no	yes	yes	27	25	28	31
8	no	no	yes	15	17	14	11

Tab. 6: The best models – matrix inversion

	G^2	df	p-value	delta
fitDET	0.0	0	1.000000	0.000
fitDE.DT.ET	2.5	1	0.110956	0.038
fitDE.DT	5.0	2	0.080695	0.040
fitDT.E	11.8	3	0.008067	0.071

Tab. 7: Goodness of fit of the matrix inversion model

The models DE.DT.ET and DE.DT fit the data sufficiently. As in the case of linear dependence the conditional independence model DE.DT is the best among all conditional independence models (DE.DT, DE.ET, DT.ET); the weakest relationship is again in the pair ET. The remaining pairs DE and DT hold their (significantly) positive association and could not be omitted from the model without losing its statistical significance.

Although the joint independence model DT.E fits the data insufficiently, this model is the best among the other joint independent models. The relationship DT can be considered to be the strongest among the relationships DE, DT, ET.

The odds ratios of DT and DE were calculated from the model DE.DT (the best of the statistically significant models): let the meaning of Dy, Dn, Ey, En is as above, let Ty, Tn denote „theory yes“ (the theory was expected) and „theory no“ (the theory was not expected). Then (Dy/Dn)/(Ty/Tn) = exp(1.44) = 4.239 and the odds ratio of DE is (Dy/Dn)/(Ey/En) = exp(0.74) = 2.096.

This means that increasing the ratio Ty/Tn, the ratio Dy/Dn increases about four times. Similarly, increasing the ratio Ey/En, the ratio Dy/Dn doubles.

Evaluation of Tests in the Group C (The University of Finance and Administration)

In the academic year 2015/2016, a group of 90 students (the group *C*) of the University of Finance and Administration had the same task as the group *A* in the academic year 2009/2010 at the University of Economics in Prague (students did not expect the theoretical part in the credit test). The tests were administered and evaluated by the author of this article under the same conditions as in year 2009/2010.

Unlike the students of the University of Economics, the students of the University of Finance and Administration were exactly familiar with the structure of the credit test – they knew that the test would contain (among others) a task to calculate the rank of a matrix and determine the linear in/dependence of vectors. Almost all students have mastered these exercises (regardless of the numerical errors) – the percentage of failed students was negligible. For this reason the relationship between knowledge of definitions and the ability to solve exercises was not tested in this group. Students also knew that the test wouldn´t contain any exercise using matrix inversion – that is why students can be expected not to have studied this topic at all. As we can see in Table 8, the score corresponds to this expectations.

Table 8 shows the number of correct answers in group of 90 students of the University of Finance and Administration.

	number	%
linear combination	28	31.11%
linear dependence	39	43.33%
matrix rank	45	50%
invertible matrix	18	20%
matrix inversion	13	14.44%

Tab. 8: Success rate in formulating particular terms (the group *C*)

The success rate is significantly lower in the case of invertible matrix and matrix inversion in line with our expectations. As we can see by comparison with Table 1, this result is quite opposite to the result in the group *A* (for the reasons mentioned above).

The success rates in the groups *A* and *C* are compared by hypothesis testing (see (1) – (3)) at the 5% significance level for each of these terms. The results are summarized in Table 9.

	H_0	H_1	u	$W_{0.05}$
linear combination	$\pi_1 = \pi_2$	$\pi_1 \neq \pi_2$	- 0.019	$\|u\| \geq 1.960$
linear dependence	$\pi_1 = \pi_2$	$\pi_1 < \pi_2$	- 2.421	$u \leq - 1.645$
matrix rank	$\pi_1 = \pi_2$	$\pi_1 > \pi_2$	2.032	$u \geq 1.645$
invertible matrix	$\pi_1 = \pi_2$	$\pi_1 > \pi_2$	8.127	$u \geq 1.645$
matrix inversion	$\pi_1 = \pi_2$	$\pi_1 > \pi_2$	3.204	$u \geq 1.645$

Tab. 9: Hypothesis testing (comparison of the groups *A* and *C*)

In the case of linear combination, the calculated value of test criterion (u = - 0.019) lies outside the critical range $W_{0.05}$ – we do not reject the null hypothesis H_0: $\pi_1 = \pi_2$. The difference is not statistically proved at the 5% significance level and the similar success in both groups *A* and *C* cannot be excluded.

In the case of linear dependence, the calculated value of test criterion (u = - 2.421) lies in the critical range $W_{0.05}$ – we reject the null hypothesis in favor of the alternative hypothesis H_1: $\pi_1 < \pi_2$. It is statistically proved at the 5% significance level that students in the group *C* were more successful than students in the group *A*.

In the cases of matrix rank, invertible matrix and matrix inversion the calculated values of test criterion (u = 2.032, u = 8.127, u = 3.204, respectively) lie in the corresponding critical ranges $W_{0.05}$ – we reject the null hypothesis in favor of the alternative hypothesis H_1: $\pi_1 > \pi_2$. It is statistically proved at the 5% significance level that students in the group *A* were more successful than students in the group *C*.

As well as in the group *A*, the relationship between knowledge of the term rank of a matrix and invertible matrix, respectively, and the terms linear dependence and linear combination was examined. As we can see in Table 10, 45 students (out of 90 total) defined correctly the term rank of a matrix but only 15 (i.e. 33.33%) of them defined correctly the remaining terms. Only 9 (i.e. 50%) out of 18 answers was correct in the case of the invertible matrix.

MATRIX RANK yes **45**				INVERTIBLE MATRIX yes **18**			
Linear dependence yes **28**		Linear dependence no 17		Linear dependence yes **12**		Linear dependence no 6	
Lin. comb. yes **15**	Lin. comb. no 13	Lin. comb. yes 2	Lin. comb. no 15	Lin. comb. yes **9**	Lin. comb. no 3	Lin. comb. yes 0	Lin. comb. no 6

Tab. 10: Definitions (matrix rank, invertible matrix)

The success rates in the groups *A* (41 of 186, i.e. 22%) and *C* (15 of 45, i.e. 33.33%) are compared by hypothesis testing for the case of matrix rank. The null hypothesis H_0: $\pi_1 = \pi_2$ is tested at the 5% significance level against the alternative hypothesis H_1: $\pi_1 < \pi_2$. The calculated value of test criterion (u = - 1.586) lies outside the critical range $W_{0.05} = \{u;\ u \leq - 1.645\}$ – the *alternative* hypothesis H_1: $\pi_1 < \pi_2$ cannot be accepted. It failed to statistically prove at the 5% significance level that students in the group *C* were more successful than students in the group *A*. However, it can be proved at the 10% significance level (even at the 6% level).

The number of correct answers is too low for statistical evaluation in the case of the invertible matrix.

Students in the group *C* seem to be able "to go to the root of the matter" better than students in the group *A*, but this fact is statistically proved only at the 10% significance level.

Most common errors

Most common errors were evaluated and compared in the groups *A* and *C*. In the case of definition of linear dependence and matrix inversion, also the results of the group *B* were taken in account.

The most common errors have the same character in all groups, regardless whether the students studied the theory or not. In general, in most cases the students were not able to formulate terms in generic way, struggled with generalized notation, ignored quantification, did not make any differences between definitions and theorems. Formulations often did not make any sense at all. The definitions correctly formulated by students were very often formulated identically with wording and notation as in the textbook, in particular in groups *A* and *B*. Only in very sporadic cases the students managed to formulate the definition correctly in his/her own words.

Especially for each of tested definitions the most common errors are identical. Tables 11 - 15 show numbers of correct answers, most common errors, other errors and unanswered tasks (no attempt to formulate the definition). As we can see in these tables, the percentage of blank answers is much greater in the group *C* than in the group *A*. Except data in the tables, the number of most common wrong answers is related to the number of all incorrectly answered questions (except the blank answers). However, the low number of incorrect answers (and high number of blank answers) in the group *C* is not sufficiently conclusive in some cases for a statistical comparison of both groups in this respect. Nevertheless, the similarity of the character of most common error in both groups *A* and *C* is significant – not the fact that the percentage of these answers is higher or lower in the group *A* compared with the group *C*.

In the case of linear combination, the formulation of the sense “linear combination of the vectors are their multiples” is the most common error. Such formulation takes about 45% (49 of 108) of all wrong answers in the group *A* and 32% (8 of 25) in the group *C*. Detailed score is shown in Table 11.

Linear combination	*A*		*C*	
	number	%	number	%
correctly	93	31	28	31.11
multiple	49	16.33	8	8.89
other error	59	19.67	17	18.89
nothing	99	33	37	41.11
total	300	100	90	100

Tab. 11: Most common errors (linear combination)

In the case of linear dependence, the formulation of the sense “the vectors are linear dependent when one of these vectors is a multiple of some of others” is the most common error. Such formulation takes about 58% (102 of 175) of all wrong answers in the group *A* and 43% (10 of 23) in the group *C*. Detailed score is shown in Table 12.

Linear dependence	*A*		*C*	
	number	%	number	%
correctly	89	29.67	39	43.33
multiple	102	34	10	11.11
other error	73	24.33	13	14.44
nothing	36	12	28	31.11
total	300	100	90	100

Tab. 12: Most common errors (linear dependence)

In the case of matrix rank, the formulation of the sense “the rank of matrix is the number of (nonzero) rows” is the most common error. Such formulation takes about 64% (63 of 98) of all wrong answers in the group *A* and 80% (24 of 30) in the group *C*. Detailed score is shown in Table 13.

Matrix rank	*A*		*C*	
	number	%	number	%
correctly	186	62	45	50
nonzero rows	63	21	24	26.67
other error	35	11.67	6	6.67
nothing	16	5.33	15	16.67
total	300	100	90	100

Tab. 13: Most common errors (matrix rank)

In the case of invertible matrix, the statement “the matrix is invertible, if its determinant is nonzero“ is the most common error (it is true, but in curriculum the definition is presented using rank of the matrix, not determinant – using a theorem instead of a definition is considered to be an mistake at the University of Economics). Such formulation takes about 46% (31 of 67) of all wrong answers in the group *A* and 43% (10 of 23) in the group *C*. Detailed score is shown in Table 14.

Invertible matrix	*A*		*C*	
	number	%	number	%
correctly	205	68.33	18	20
det A nonzero	31	10.33	10	11.11
other error	36	12	13	14.44
nothing	28	9.33	49	54.44
total	300	100	90	100

Tab. 14: Most common errors (invertible matrix)

In the case of matrix inversion, the most common errors are: a description of the procedure of the calculation and the formulation “invertible matrix is the matrix opposite to the given matrix” (without any specification). The first type of wrong formulation takes about 35% (55 of 157) of all wrong answers in the group *A* and 31% (13 of 42) in the group *C*, the second type takes about 22% (34 of 157) of all wrong answers in the group *A* and 45% (19 of 42) in the group *C*. Detailed score is shown in Table 15.

Matrix inversion	*A*		*C*	
	number	%	number	%
correctly	95	31.66	13	14.44
procedure	55	18.33	13	14.44
opposite	34	11.33	19	21.11
other error	68	22.67	10	11.11
nothing	48	16	35	38.89
total	300	100	90	100

Tab. 15: Most common errors (matrix inversion)

The even more detailed list of errors and numerous samples of formulations in the groups *A* and *B* are in Ulrychová (2013). Some remarks to the method of teaching the problematic terms are in Ulrychová (2013) and Ulrychová (2014).

Discussion

The results of the tests, their statistical evaluation and the analysis of the common errors lead to the following conclusions.

The statistical evaluation of results in the groups *A* and *B* gives an interesting result. One can expect the relationship between learning the theory and the correctness of the definition (DT) to be the strongest among the tested relationships (DT, DE, ET). In the case of matrix inversion, the result is in agreement with this expectation, but in the case of linear dependence, the relationship between correctness of the definition and the exercise is the strongest. That means that the level of students’ general mathematical skills is more crucial than the study of the theory. That may be caused by the fact that students find the definition of linear dependence rather difficult to understand and formulate. On the other hand, the definition of matrix inversion is easy to understand and remember. This fact also corresponds with the results of the tests – the correctness of definition of matrix inversion doubled by learning the theory (from 33% to about 68%), whereas the correctness of definition of linear dependence increased much less (from 36% to 42%).

The statistical result in both cases (linear dependence and matrix inversion) shows that the relationship ET is the weakest; i.e. the correctness of exercise does not depend on learning the theory. This means that the ability to solve exercises did not increase by learning the theory. The independence of the skills to solve exercises on the ability to formulate definitions is also confirmed by the results in the group *C*, in which almost all students solved right the exercises, but many of them did not even attempt to formulate definitions. On the other hand, the success rate of formulations of terms related to exercises that students had not expected in the test was even lower – it seems that the students may nevertheless have studied the theory when preparing for solving the exercises.

All the groups show the same character of the most common errors, irrespectively of whether students studied the theory or not and regardless of the type of the school. In particular, the most common errors are identical for each of the tested definitions.

However, the acquaintance of a definition still does not mean the knowledge of the matter. As shown in Table 2 and Table 10, the students formulated a given term correctly based on other terms, which they could not formulate correctly. The importance of such knowledge is then questionable.

There is a question: is it beneficial to ask students to formulate definitions exactly, when there is not enough time to practice it? For example, students can calculate the rank of matrix and use it for decision whether a system of linear equations has a solution or not – is it really necessary for students to be able to formulate the definition of rank of matrix and all terms related to? The teachers of specialized courses at universities of economic studies are generally consistent in the opinion that students should to master the calculus rather than theory. On the other hand, the teachers of mathematics are not consistent in this opinion.

There are some essential disadvantages of written form for testing the ability to formulate definitions and theorems. It is impossible to determine with certainty whether the student memorized the definition (without understanding it) or not. Moreover, the teacher cannot gradually correct the errors and help the student to reach the correct expression as in the case of oral examination. During the oral examination the teacher is able to differentiate if the student does not understand the term at all or if he/she has a pretty good understanding about the term but is just not able to express it. The latter case is certainly more beneficial than the case if the student just memorizes the terms without understanding them.

Although students' performance could among others depend on various factors – e.g. on the field of their study (Otavová and Sýkorová, 2015) or on the person of the teacher and his/her teaching methods (Majovská, 2015), (Milková and Kořínek, 2014), (Widenská, 2014), the errors, that the students make, have the same character independent on teacher and university (students of four teachers at two universities were tested). The analysis of most common errors can contribute to the improvement of the way of interpretation and practice (Matulová, 2015), if there is time for it.

Conclusion

In the case of the University of Economics in Prague with one-semester course of mathematics with a relatively wide curriculum, it is impossible to exercise the students to make them able to formulate mathematical terms precisely. The students solve this problem by memorizing definitions and they are not able to interpret results obtained by calculations. It is questionable whether in such case one should insist on precise wording of the definitions or to be satisfied at least with a general idea and to prefer the knowledge of relationships, the ability to make right conclusions and to interpret the results.

The approach to explaining the theory and namely to the oral examination is rather different at the University of Finance and Administration compared to the University of Economics. Interpreting the theory, the understanding is crucial and the exact formulations are not necessarily required from students during the oral examination – students can describe the essence in their own words with the possibility to correct themselves under the guidance of their examiner. Unfortunately, the low number of lessons does not allow students to practice the formulations of the definitions and theorems in the seminars. Another challenge is the high proportion of foreign students having difficulty speaking Czech. The rather less formal approach to teaching mathematics – compared to the University of Economics – did not yield better results in students' ability to formulate definitions, as shown above. However, according to students' poll, students appreciate this way of teaching and find it still difficult but more comprehensible.

Although the written form of testing of the ability to formulate definitions is not very appropriate for the abovementioned reasons, the most common errors are consistent with the author's experience during oral examinations at both universities (the University of Economics in Prague and the University of Finance and Administration). Also the statistical evaluation of the results of the tests is in accordance with teaching experience – the students' ability to solve exercises is often independent on study of the theory.

Students of universities of economic studies often do not accept the fact that in modern economics the position of mathematics is quite significant. This leads to their negative approach where they presume that mathematics is useless for their studies (Pražák, 2014). Teachers should try to change this approach – a comprehensible way of teaching should contribute to it. Based on the results presented in this article, it should be considered, to which extent the theory ought to be emphasized in mathematics courses at universities of economic studies. The main aim of the mathematical course at universities of economic studies is usually to make students able to use mathematical procedures for solving tasks in specialized courses. The fact that the students of these schools need primarily to master calculus rather than theory, definitely does not mean that the theory should not be taught at all. However the method of explaining the curriculum should be adequate to the specialization of the school and the teacher should consider the extent to which it is beneficial to ask students to formulate the definitions precisely (if there is no opportunity to practise it). On the other hand if the course provides enough time to practise correct formulations, students may benefit from it and apply them not only in the course of mathematics but also in other courses, as well as in their professional life (cf. Milková (2011)). Unfortunately, declining number of lessons of mathematics at some universities, as for example at the University of Economics in Prague (in detail in Ulrychová (2013)), does not allow to be much optimistic in this respect.

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