EXPLORATION OF SPATIAL ABILITIES IN PRE-SERVICE MATHEMATICS TEACHER EDUCATION: TESTING AND EVALUATION

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ABSTRACT

Pre-service mathematics teachers often struggle with spatial ability, which negatively affects their success in solving geometric problems. Evaluating and developing these abilities is therefore an essential part of their university education. This paper presents findings from the initial phase of a long-term study focused on assessing the spatial ability and conceptual knowledge of first-year pre-service teachers at Charles University. Each year from 2021 to 2023, newly enrolled students were tasked with completing tests focused on 2D and 3D geometry, classified according to specific subcomponents of spatial ability. The results show that the students were most successful in planar rotation tasks, with the tasks requiring spatial visualisation proving to be the most challenging. Conceptual misconceptions were identified as a key factor contributing to errors in solving geometric tasks. These findings highlight the need for targeted instruction and training to improve spatial thinking and conceptual understanding in teacher education, with a view to improving the quality of the geometry teaching they provide in the future.

KEYWORDS

Conceptual knowledge, efficiency in geometry education, planar and spatial geometry, pre-service mathematics teacher education, spatial ability, testing spatial abilities

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Highlights

- *Tasks involving spatial visualisation pose the most difficult challenge for pre-service mathematics teachers.*
- *Planar tasks are easier for pre-service mathematics teachers than spatial ones.*
- *Deficiencies in conceptual knowledge contribute in varying degrees to errors in spatial geometry tasks undertaken by students.*

INTRODUCTION

Many students, including future mathematics teachers, face persistent difficulties when solving geometric problems. Research has repeatedly shown that geometric tasks in a three-dimensional space tend to be more demanding than planar tasks (Ismail and Rahman, 2017). A broader study of 1,357 students (Grades 4–9) found that even combined spatial abilities were insufficient to handle problems that required more than one step of spatial reasoning. Students' success depended on the integration of domain-specific geometric knowledge, such as knowledge of the elements, properties and concepts of geometric figures (Fujita et al., 2020). Other studies have also shown that difficulties in geometry are often related to underdeveloped spatial abilities (Sorby and Panther, 2020) or limited conceptual understanding (Rittle-Johnson and Schneider, 2015). It is important to address students' difficulties with geometry learning during their university education—especially in the case of future mathematics teachers.

Theoretical Framework

The success in solving geometric problems critically depends on spatial ability. Lohman (1979) describes *spatial ability* as the ability to generate, retain, retrieve and transform well-structured visual images. Lean and Clements (1981) define spatial ability as the ability to formulate mental images and to manipulate these images in the mind. A similar definition is presented by Linn and Petersen (1985), who state that spatial ability generally refers to the skill of representing, transforming, generating and recalling symbolic, non-linguistic information. Sorby (1999) makes a distinction between *spatial ability* and *spatial skills* – spatial ability is considered an innate capability for visualisation, while spatial skills are acquired through training and learning.

Nevertheless, these terms are closely interconnected; it is difficult to distinguish between them. In line with literature, the term spatial ability is used uniformly here, with the way in which the ability was acquired, not examined.

Extensive research on spatial ability has led to the development of detailed categorisations of its subcomponents. These categorisations have also been modified and expanded over time. However, there is no clear and consistent model for these subcomponents of spatial ability. For example, McGee (1979) described two major subcomponents (factors in his terminology) of spatial ability – *spatial visualisation* and *spatial orientation*. This categorisation is based on the mental processes used for solving certain tasks. Lohman's (1979) classification consists of three basic spatial ability subcomponents – *spatial relation*, *spatial orientation* and *visualisation*. Carroll (1993) identified five major subcomponents of spatial ability – *visualisation*, *spatial relations*, *closure speed*, *flexibility of closure* and *perceptual speed*. The number of underlying subcomponents of spatial ability seems to vary across studies. However, *visualisation*, *spatial relation*, *mental rotation* and *spatial orientation* are commonly recognised as relevant subcomponents of spatial ability in current research (Maresch and Posamentier, 2019). These subcomponents are considered in the presented research because they best suit the types of geometric tasks applied. In the following text, each subcomponent of spatial ability is described. We also provide examples of typical tasks in which it is used.

The subcomponent *visualisation* is usually described as a general subcomponent of spatial ability. This subcomponent is defined as the ability to think of changes in objects, changes in position, orientation, or internal relationships. This implies that we mentally manipulate or alter the imagined object or its components. According to Maresch and Posamentier (2019), this can include tasks such as mentally folding objects from their nets and vice versa, identifying the opposite sides of a cube from its net, folding a piece of paper and cutting it mentally, completing cubic nets, drawing a net of a solid figure with patterns on its sides from its 3D model, and so on.

Spatial relation is a crucial subcomponent of spatial ability when assembling and organising objects in both two-dimensional and three-dimensional spaces. This subcomponent involves understanding how various parts of an object fit together and how these parts relate to each other in a three-dimensional space. This ability is essential for tasks that require comparing various objects, mentally manipulating and assembling them, and forming a complete structure from separate elements. Spatial relation shares some common features with the visualisation subcomponent but demands a more specific kind of mental manipulation, with emphasis placed on the arrangement and interaction of parts within a whole object. Typical tasks, which can be included here, are the problem of packing luggage (figuring out the most efficient arrangement of items), finding the matching parts from shown structures which can be used to fill another given structure, cutting objects into two parts, building models from small cubes, which involves calculating the number of cubes needed, and so on.

Mental rotation is a subcomponent of spatial ability involving the ability to imagine the rotation of both two-dimensional and three-dimensional objects. Mental rotation tasks typically require the identification of geometric objects, often presented in various positions, and their mental rotation. A common challenge is determining whether two rotated objects are identical (the classical Mental Rotation Test), finding one different object among others, rotating an object around an axis, determining around which axis the object must be rotated to get to the new position, and so on. Furthermore, these tasks usually test the speed with which the problem is solved.

Spatial orientation is a subcomponent of spatial ability which is required for mental orientation in a three-dimensional space. This ability involves understanding and mentally moving around a spatial arrangement of objects. This aspect requires individuals to imagine an object's appearance from various viewpoints. This means that instead of moving the objects in our minds, we mentally shift our own perspective. Tasks assessing spatial orientation often include determining the viewpoints in a 3D map, figuring out from which direction relative to the initial one we observe an object, and given a top view of a parking lot with labels, deciding how these labels would appear when viewed from different perspectives.

Currently, researchers also focus on the identification and description of strategies for solving geometric problems. Traditional research methods regarding subcomponents of spatial ability typically assume that tasks within a specific category are solved using the same intended strategy. However, based on the findings in literature (Maresch and Posamentier, 2019; Kozhevnikov and Hegarty, 2001) and our own experience working with students, it is evident that geometric tasks are approached differently by individuals. We designed a test with geometric tasks, keeping in mind our aim to assess specific subcomponents of spatial ability. While these tasks were primarily designed to test specific subcomponents of spatial ability, dividing them into strictly defined categories can prove challenging. For example, it is possible that the categories overlap, meaning a single task may assess more than one subcomponent of spatial ability. Consequently, the boundaries between task categories cannot distinctly be set, reflecting the individualised approaches of students in problem-solving, as is highlighted in other research publications (Carroll, 1993; Kozhevnikov and Hegarty, 2001). This realisation underscores the complexity of spatial ability assessment, revealing that while tasks are designed with specific spatial ability in mind, they often intersect across multiple spatial ability domains.

Spatial ability is a crucial aspect of intellectual ability. Research has shown that regular training can significantly strengthen this skill. A number of scientists support the idea that targeted interventions can improve spatial ability and have explored effective methods for its development (Gold et al., 2018; Lowrie et al., 2019; Prieto and Velasco, 2010; Šafhalter et al., 2022; Sorby and Baartmans, 2000).

Spatial ability and its subcomponents are usually evaluated through standardised tests. These tests typically focus on a specific aspect of spatial ability, requiring participants to solve similar tasks that vary in complexity. In certain tests, researchers not only evaluate the accuracy of participants' responses but also take into account the speed at which they answer. One well-known test is, for example, the *Mental Rotations Test* (Vandenberg and Kuse, 1978) or its redrawn, modified version, called MRT-A (Peters et al., 1995). These tests assess the mental rotation component of spatial ability. Although other tests exist, none of them were suitable for our research. However, they provided inspiration for developing a completely new test for our students. The primary aim was to assess not just one, but all four defined subcomponents of spatial ability, while also taking into account the important role of conceptual and procedural understanding in mathematics education. The test therefore not only assesses spatial ability, but also evaluates students' understanding of mathematical concepts and their ability to apply procedural knowledge. A number of research studies have shown a link between performance on tests of spatial imagination and mathematical achievement (Cheng and Mix, 2014; Harris, 2021; Resnick et al., 2020; Sorby and Panther, 2020).

Effective evaluation requires assessing students' spatial abilities as well as their understanding of concepts and procedures. Conceptual and procedural knowledge are considered two key cognitive principles in mathematics. The first is usually defined as 'comprehension of mathematical concepts, operations, and relations' (Kilpatrick et al., 2001: 5), or simply as 'knowledge of concepts', because 'more recent thinking views the richness of connections as a feature of conceptual knowledge that increases with expertise' (Rittle-Johnson and Schneider, 2015: 1119). Procedural knowledge is understood as 'the ability to execute action sequences (i.e., procedures) to solve problems' (Rittle-Johnson and Schneider, 2015: 1120).

Conceptual and procedural knowledge are closely related. While many concepts in the field of arithmetic arise from mathematical processes (Dienes, 1967), in geometry, the child first perceives the concept and procedural knowledge follows (Hejný, 2000). Researchers generally agree that the development of conceptual knowledge improves procedural knowledge rather than vice versa (Hecht and Vagi, 2010; Rittle-Johnson and Schneider, 2015; Rittle-Johnson et al., 2015; Rittle-Johnson et al., 2001; Star, 2005). Furthermore, according to Son (2006), pre-service teachers have limited conceptual knowledge in the field of geometry and tend to rely on procedural knowledge.

Many studies have found an association between spatial ability and mathematical ability (e.g., Sorby and Panther, 2020; Young et al., 2018). On the other hand, Xie et al. (2020) point out that an increasing number of research studies demonstrates that associations between spatial and mathematical ability may not be consistent across all spatial and mathematical components. They analysed studies published in 2008–2018, investigating the relationship between spatial and mathematical abilities. They did not prove a causal relationship between these abilities, but suggested that logical reasoning was more strongly associated with spatial ability than numerical and arithmetical ability.

It is generally known that 3D geometry problems tend to be more difficult for students than 2D geometry problems. This is confirmed, for example, by a study conducted by Ismail and Rahman (2017). This study found significant differences in the examination of 2D and 3D formations at the level of analysis and informal deductive reasoning among students who used GeoGebra. These students were more successful with 2D shapes.

The difficulty of 2D and 3D tasks for students may not only depend on the level of their spatial ability but may also be related to the formulation of the tasks or their representation. For example, solving a problem in a 3D computer environment can be easier than in a 2D environment, as the student can visualise the spatial situation from different points of view.

Research Aims and Questions

While both spatial ability (including its subcomponents) and conceptual knowledge have been extensively studied in mathematics education, they are usually analysed separately. Our study introduces a novel perspective by combining these two dimensions in the analysis of specific geometric task types: we identify which subcomponent of spatial ability is required for particular task types, we examine how deficiencies in conceptual knowledge influence students' success or failure, and we apply this approach in the university setting of pre-service mathematics teacher education. This approach offers valuable new insights into students' difficulties in geometry. The findings can help improve university geometry courses by showing which topics students struggle with and need more support in.

Within this context, our research aims to assess Czech students' proficiency in solving diverse geometric problems. To this end, both planar and spatial tasks were included to examine students' performance across these domains. While it is generally known that students perform better in planar geometry than in spatial geometry, our study goes beyond this general comparison by analysing students' success in specific task types targeting different subcomponents of spatial ability and by examining how deficiencies in conceptual knowledge contribute to errors. This detailed analysis enables the determination of what tasks are more difficult. Based on the students' results, courses on geometry, which students attend during their university studies, can be modified. Specifically, the study targets pre-service mathematics teachers at the Faculty of Mathematics and Physics, Charles University.

In line with the above, the following research questions were formulated:

- RQ1: What is the students' success rate in individual geometric tasks (also with consideration for differences between men and women)?
- RQ2: In which type of geometric tasks targeting various subcomponents of spatial ability do students perform best/worst?
- RQ3: Are students more successful in the planar or in the spatial geometric tasks?
- RQ4: How do deficiencies in conceptual knowledge contribute to students' errors in understanding tasks?

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The paper is organised as follows. The Materials and Methods section outlines the test used to assess spatial ability among students, including a detailed explanation of the research methodology. This is followed by the Results section, which presents task success rates, compares male and female performance, examines the different subcomponents of spatial ability, and contrasts planar versus spatial tasks. Furthermore, the impact of conceptual knowledge on performance in geometric tasks is examined. The Discussion section then contextualises the findings within the framework of international studies and addresses each research question individually. The paper concludes with a concise summary, suggestions and ideas for future work.

MATERIALS AND METHODS

Research Context and Participants

Throughout the period of the research, the primary focus was on testing the spatial abilities of pre-service mathematics teachers at the beginning and end of their university studies. In this paper, only the results from first-year students are presented, as these participants have not finished their studies yet. The aim is to analyse students' success in geometric tasks designed to evaluate specific subcomponents of spatial ability alongside their conceptual understanding. A key component of the assessment also involves an evaluation of students' conceptual knowledge in order to provide a more comprehensive understanding of their proficiency in geometry.

Year	Number of students (women/men)	Test format	Time limit (standard/extended)	Number of tasks
2020 (pilot)	36 (25/11)	online	30/40	22
2021	36 (17/19)	in-person	30	26
2022	25 (10/15)	in-person	30	26
2023	25 (11/14)	in-person	30	26

Table 1: General characteristics of the testing, 2020–2023 (source: own data)

The students who participated in the test were pre-service mathematics teachers in the first year of their university studies, i.e. newcomers to the faculty (Faculty of Mathematics and Physics, Charles University, Czech Republic). The testing was repeated four times: in the years 2020 (pilot study, Surynková et al., 2021), 2021, 2022 and 2023, always with different groups of students. The general characteristics of the testing are summarised in Table 1.

Test Design and Development

The geometric problems in the test are designed so that a specific spatial ability is tested by solving them. Each task also assesses certain conceptual knowledge in geometry. In the first year of testing (i.e. 2020), the test consisted of 22 individual geometric tasks. After analysing this initial test, which served as a pilot version, the authors revised it to better balance the complexity of the tasks, aiming to enhance the graphical clarity and comprehensibility of the task descriptions. An additional task focused on planar geometry was also added to further diversify the range of problems. Compared to the pilot version, four tasks were added and four were modified. As a result, in the subsequent years (2021, 2022 and 2023), the test consisted of 26 individual geometric tasks. The tasks are numbered from 1 to 14, with some divided into related subtasks, making a total of 26 tasks. When describing the tasks, the notation Task 12.1, for example, refers directly to a specific subtask, while the notation Task 12, for example, indicates that the results pertain to all its subtasks.

Task Categorisation

A brief description of the tasks from the final version of the test from years 2021, 2022 and 2023 is presented in Table 2, where, among other data, the individual task success rate across the years is provided (column **M**). The tasks are primarily divided into two groups – **2D** (two-dimensional tasks) and **3D** (three-dimensional tasks). The test includes four types of tasks focused on subcomponents of spatial ability: visualisation (**V**), spatial relation (**SR**), mental rotation (**MR**) and spatial orientation (**SO**). The categorisation of the tasks into these types was thoroughly discussed within the team, drawing upon professional literature and the team members' experience. It is important to note that some tasks may fall into multiple categories or may not be typical for a given category (denoted by a dot in brackets); their inclusion is based on professional judgement (the authors of the test have dedicated themselves to teaching geometry and training spatial skills for many years) and the specific objectives of the test. All the tasks test properties of elementary objects in a plane and in a space such as a straight line, circle, solid figure, etc. Other specific conceptual knowledge areas are listed in Table 2.

Procedure

The test begins with preliminary questions about the participant, including the student's name, age and study specialisation. These data indicate the fundamental characteristics of the students who complete the test. The test is not anonymous because the research is planned as long-term research, i.e. The same group of students will be tested in the future again. All students were informed about the objectives and long-term nature of the testing.

The standard time limit for the test was 30 minutes; the extended time limit for students with special needs was 40 minutes. The students solved the test individually during regular courses.

The students were also asked for their opinion on the difficulty of the test and the strategy they used to solve the individual tasks. The students wrote these comments voluntarily. A comprehensive analysis of these comments was not undertaken. Instead, we only examine comments relating to specific tasks, which are discussed in the Discussion section.

Data Analysis

The acquired data were processed both quantitatively and qualitatively. The absolute and relative frequencies of students' answers were determined, with certain dependencies observed. We compared the results of students in individual tasks, across the subcomponents of spatial ability, and also examined the performance differences between men and women. Furthermore, we analysed the outcomes of tasks related to planar versus spatial geometry and investigated the impact of conceptual knowledge on the results. The statistical significance of dependencies between the obtained data was examined by applying Pearson's χ^2 test (which could only be used for some subtasks) and Fisher's exact test (always used due to the conditions not being met for the χ^2 test). To evaluate student comments, we employed qualitative methods and categorised the comments into groups for further analysis.

	No.	M	V	SR	MR	SO	Tested conceptual knowledge	Task characteristics
2D	5.1	91.86			•[1]		- angle of rotation - sign of an angle - rotation around a point - identification of the size of an angle in the square grid	rotating objects around a point in a plane
	5.2	86.05			•[1]			
	5.3	77.91			•[1]			
	5.4	84.88			•[1]			
	8.1	96.51		•			- properties of a circle - number of common points (intersections) of two circles - external and internal tangency of two circles - relationship between radii and circle centre distances	the relative positions between two circles in a plane (solved without pictures)
	8.2	95.35		•				
	8.3	63.95		•				
	9.1	69.77		•			- properties of a circle - number of common points (intersections) of two circles - external and internal tangency of two circles - relationship between radii and circle centre distances - the notation and the meaning of conjunction	the relative positions between two circles in a plane (solved without pictures)
	9.2	70.93		•				
	9.3	84.88		•				
	12.1	97.67			•		- rotation around a point - direct and indirect congruence	rotating objects around a point in a plane
	12.2	98.84			•			
	12.3	100.00			•			
	12.4	100.00			•			
3D	1	84.88		(•)		•	- determination of a plane - intersection of two planes - intersection of a line and a plane - determination of the sides of a cut (two points, parallel lines)	cross-section of solids
	2	90.70		(•)		•	- determination of a plane - intersection of two planes - intersection of a line and a plane - determination of the sides of a cut (two points, parallel lines)	cross-section of solids
	3.1	52.33	•				- determination of a plane - distinguishing of positional and metric properties - parallel lines - skew lines - intersecting lines	the relative positions between two lines in the space
	3.2	70.93	•					
	3.3	60.47	•					
	4	80.23			•		- rotation around a cube edge - skew lines	finding rotated object among others in the space
	6	98.84		•			- measurement of planar objects	assembling cut objects into parts in the space
	7	95.35		•			- measurement of planar objects	assembling cut objects into parts in the space
	10	91.86	•				- top, front and side view of an object - visibility of solid figure edges	identifying the object from top, front and side view
	11	79.07	•				- top, front and side view of an object - visibility of solid figure edges	identifying the object from top, front and side view
	13	89.54			•		- rotation around an axis - top view of an object - Cartesian coordinate system	rotating objects around axis in the space and its projection into a plane
	14	70.93			•		- rotation around an axis - top view of an object - Cartesian coordinate system - composition of transformations	rotating objects around axis in the space and its projection into a plane

[1]*Atypical mental rotation task because the students were asked to determine the size of an angle and its sign.*

Table 2: Categorisation of geometric tasks with brief description, 2021–2023 (source: own data)

RESULTS

Success Rate of Individual Tasks

Initially, the individual task success rate was analysed. Subtask 3.1 was performed the worst, with an average success rate of 52.33% across all tested years (2021–2023). Students also performed poorly on Subtask 3.3, which had an average success rate of 60.47% across all years. Other notable tasks include 8.3, with a 63.95% success rate, and 9.1, with a 69.77% success rate. Subtasks 12.1, 12.2, 12.3 and 12.4 were solved by students almost flawlessly (respectively, 97.67%, 98.84%, 100.00%, 100.00%). Tasks 6 and 7 were also performed well, with success rates of 98.84% and 95.35%, respectively. Subtasks 8.1 and 8.2 were also solved relatively well across all years, with success rates of 96.51% and 95.35%, respectively. Interestingly, Subtask 8.3 had a significantly lower score of 63.95% despite being of the same type.

An analysis was also undertaken of the tasks performed consistently over the years, i.e. those tasks students performed either consistently successfully or consistently unsuccessfully over the years. This was determined by identifying the smallest standard deviations from the average success rate of a task over the years. Consistently successful subtasks were 12.3 ($\sigma = 0\%$), 12.4 (0%), 12.2 (1.33%), 12.1 (1.68%) and 9.3 (2.07%). For the least successful tasks, the low scores were not consistent across all years. This means that a task in which students made errors in one year was solved successfully in other years. For example, Subtask 3.3 had a success rate of 48.00% in 2022, but improved to 68.00% in 2023. Likewise, Subtask 9.2 had a success rate of 58.33% in 2021, whereas in 2022 and 2023, it was 80.00%.

Men and Women

For all tested years, the average score of the men was slightly better than that of the women, as shown in Table 3. The χ^2 test and Fisher's exact test (which was used as a control) confirmed that, with few exceptions, these differences were not statistically significant (at the 5% level of significance). In the analysis of the success rates of men and women in individual tasks, statistically significant differences in favour of men were found only in Subtasks 8.3 ($p = 0.0236$) and 9.1 ($p = 0.0373$) across all years (both tasks are among those with the lowest scores).

	Men	Women	Overall
2021	81.58	77.60	79.70
2022	87.44	85.38	86.61
2023	88.46	86.36	87.54
2021–2023	85.33	82.18	83.94

Table 3: Average scores of all participants by gender and year (relative frequencies expressed as percentages), 2021–2023 (source: own data)

SUBCOMPONENTS OF SPATIAL ABILITY

Some tasks fall into multiple categories (see Table 2). These tasks were statistically included in all considered categories. The least successful was the subcomponent of spatial representation *visualisation*, where the average success rate was 70.93%. Across the years, this did not vary significantly. Other categories were comparable, with *spatial relation* at 85.12%, *spatial orientation* at 87.79%, and *mental rotation* at 88.90%. A summary is presented in Table 4.

Year	V	SR	MR	SO
2021	68.33	79.72	84.85	81.94
2022	72.80	88.80	90.91	90.00
2023	72.80	89.20	92.73	94.00
2021–2023	70.93	85.12	88.90	87.79

Table 4: Success rate of tasks by year according to the subcomponents of spatial ability (relative frequencies expressed as percentage), 2021–2023 (source: own data)

Planar and Spatial Tasks

The success rate of planar versus spatial tasks across the years was also examined. A consistently higher success rate in planar tasks was observed compared to spatial tasks in each of the years 2021, 2022 and 2023, and on average across the years (see Table 5).

	Planar tasks	Spatial tasks
2021	81.94	77.08
2022	90.57	82.00
2023	90.86	83.67
2021–2023	87.04	80.43

Table 5: Success rate of planar and spatial task by year (relative frequencies expressed as percentage), 2021–2023 (source: own data)

Fourteen of the 26 tasks in the test focused on exploring spatial ability in a plane. These were Subtasks 5.1–5.4, 8.1–8.3, 9.1–9.3 and 12.1–12.4 (see Table 2).

The higher success rate of students in the planar tasks is also evidenced by the fact that Task 12 on the rotation of letters and symbols in a plane was the most successful overall, as the average success rate in its four subtasks was 99.15% (the arithmetic mean of the success rate of four subtasks). In the other plane tasks, the average success rate over the three years of the research was as follows – 85.17% (Task 5), 85.27% (Task 8) and 75.19% (Task 9).

Influence of Conceptual Knowledge

It is assumed that the failure of students is also related to deficiencies in conceptual knowledge. The test included tasks specifically designed to assess students' understanding of key geometric concepts, not only their spatial ability. To check whether respondents were consistent in their answers, we identified subtasks that test the same conceptual knowledge and verified them using the χ^2 test and Fisher's exact test (at the 5% level of significance). This process was applied to Tasks 5, 8 and 9. It was found that, for example, associations between correct answers in Subtasks 8.3 and 9.2 were statistically significant ($p = 0.0058$). A somewhat lesser dependency was also confirmed for Subtasks 8.2 and 9.1 ($p = 0.0805$). The deficiencies in conceptual knowledge are addressed in more detail in the Discussion section, where, in addition to the mentioned tasks, Task 3 is also discussed.

DISCUSSION

It can be generally observed that performance was significantly weaker in 2021 compared to the other two years, which were relatively comparable. This raises the question of whether the impact of the COVID-19 pandemic might have contributed to these differences (Betthäuser et al., 2023; Moliner and Alegre, 2022). The shift from traditional classroom settings to remote learning during the pandemic could have affected students' cognitive abilities. However, the influence of COVID-19 on educational outcomes is not further investigated in this paper.

RQ1: What is the students' success rate in individual geometric tasks?

In our research, the success rate for individual tasks for all students ranged from 52% to 100%. The problem tasks caused difficulties for more or less all monitored years, but sometimes there were more significant differences between years.

In the most successful task (Task 12), students were required to identify from eight options the image that is not directly congruent with the given one (see Figure 1 in which Subtask 12.2 is depicted). Other tasks that were performed well were Tasks 6 and 7.

Subtasks 8.1 and 8.2 were also among the planar ones that were successfully performed. However, Subtask 8.3, which is similar, was performed one of the worst. More information about Subtasks 8.1, 8.2 and 8.3 is provided under the discussion relating to RQ4. The task that was performed worst of all was Subtask 3.1, which involved determining the relative positions of straight lines in a 3D space (see also discussion relating to RQ4).

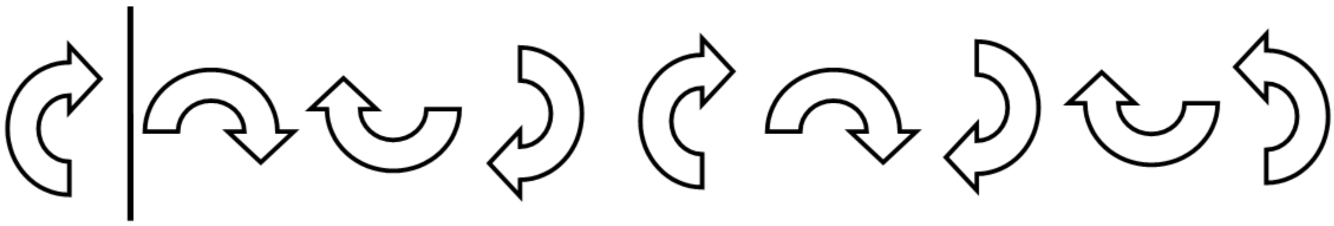

Figure 1: Task 12.2 – Identification of the drawing which is not directly congruent with the given one on the left, 2021 (source: own drawing)

The average score of men on the test was better than the average score of women across the three years of testing. With exception to Subtasks 8.3 and 9.1, these differences were not statistically significant. This result is consistent with other research (Kambilombilo and Sakala, 2015; Halat, 2008), which demonstrated no statistically significant difference between male and female pre-service mathematics teachers with reference to geometric thinking levels. Likewise, no statistically significant differences between males and females in solving geometrical tasks was observed in our research, with only a slight difference in favour of males for several tasks in the earlier research (Moravcová et al., 2021).

RQ2: In which type of geometric tasks targeting various subcomponents of spatial ability do students perform best/worst?

In our research, we tested the students' performance in geometric tasks targeting various subcomponents of spatial ability. The greatest challenge was found to be posed by tasks involving *spatial visualisation*, with an average success rate of only 70.93% over the three years. In contrast, tasks requiring *mental rotation* showed the highest success rates, at 88.9% (on average over the three years).

Tasks involving spatial visualisation required, for example, identifying the spatial object that matched the given top, front and side views in a three-dimensional coordinate system. The correct object had to be selected from four provided options (see Figure 2).

Comparing our findings with international literature reveals both similarities and differences. International research in this field often points out that spatial visualisation is a complex aspect of spatial ability (Lohman, 1979), which aligns with our observation that spatial visualisation is the most challenging for students. Lohman (1979) defined spatial visualisation tasks as complex mental transformations that are considered more difficult than other spatial tasks, which often involve simpler transformations like rotations. Spatial visualisation tasks often involve not only rotating but also manipulating objects in ways that go beyond

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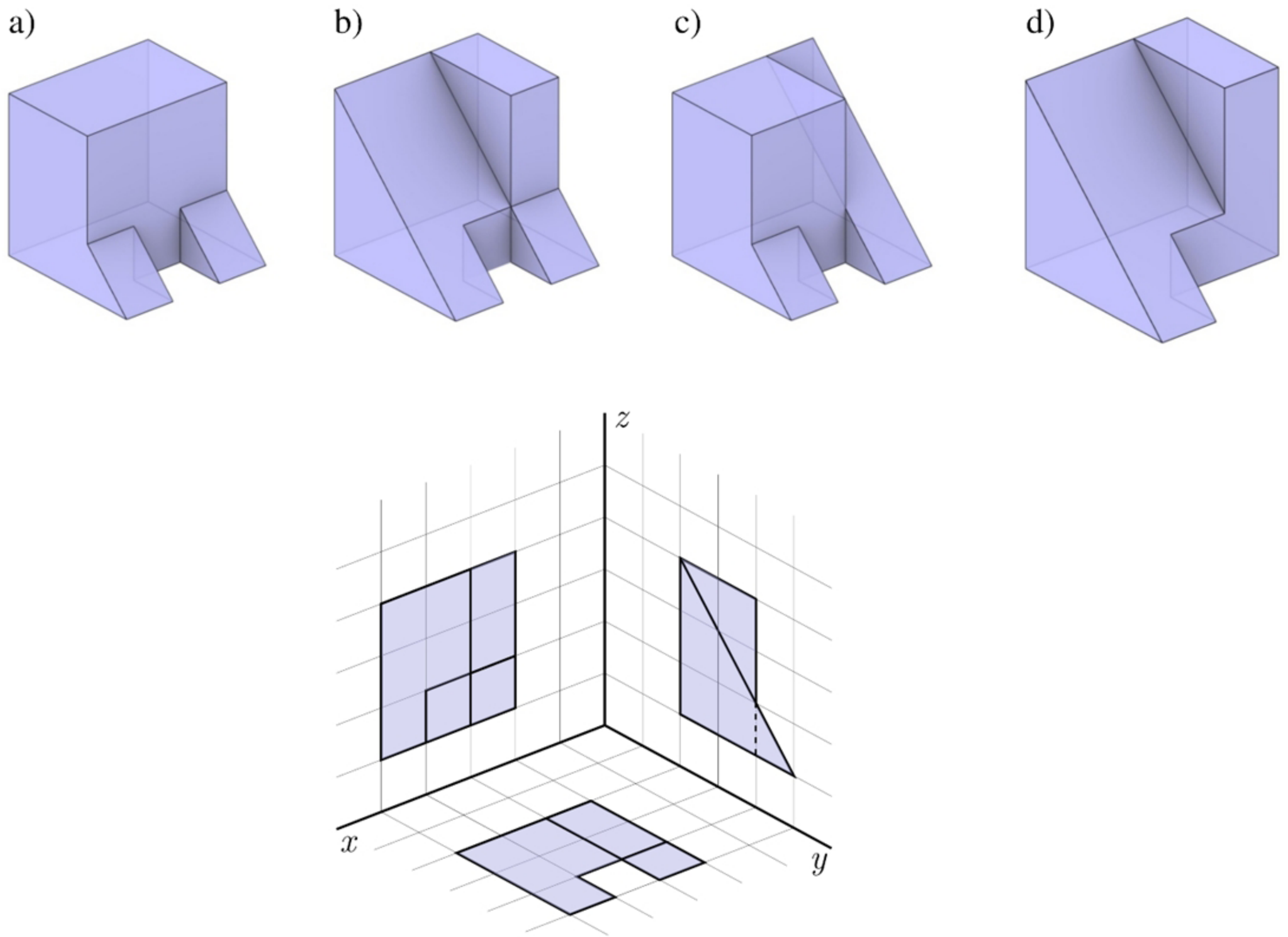

Figure 2: Task 11 – Identification of the object from top, front and side views, 2021 (source: own drawing)

simple rotation. We also observed that the success rates were higher in tasks that required less cognitive load for manipulation, as discussed by Sorby (1999).

For example, Uttal et al. (2013), in their meta-analysis, discuss the variable difficulty levels of spatial tasks among students and highlight the effectiveness of targeted training to improve specific spatial abilities. Another pivotal study by Sorby and Baartmans (2000) demonstrated the long-term benefits of spatial training for engineering students. The lower success rates of our students in visualisation tasks suggests a need for similar interventions.

In tasks involving mental rotation, important characteristics that could affect performance include the angle and axis of rotation. International literature suggests that mental rotation abilities can be assessed comparably, whether using simple cardinal-axis rotations or more complex skewed-axis rotations (Nolte et al., 2022). While our students performed well in mental rotation tasks, international studies do not suggest that mental rotation tasks are generally easier than other spatial tasks. Usually, mental rotation tasks are tested separately due to their distinct cognitive demands.

RQ3: Are students more successful in the planar or in the spatial geometric tasks?

In our research, pre-service mathematics teachers were more successful in solving 2D problems than 3D problems. Also contributing to the higher success rate of 2D problem solving is the fact that the highest scoring tasks across all three years of testing were the four planar tasks 12.1–12.4 (see Results section). However, it should be noted that some subtasks of 2D Tasks 5, 8 and 9 were performed worse. More on this is provided under the discussion relating to RQ4.

These results can be related to the fact that the Czech national mathematics curriculum for lower and upper secondary schools (MEYS, 2005; 2007) focuses more on students' outcomes in a plane than in a 3D space. This fact is also confirmed by our task analysis of state entrance tests to secondary schools and of the graduation tests for the last four years; the ratio between the number of planar and spatial tasks is approximately 3:1.

The better results of pre-service teachers in 2D tasks correspond to international research results with other respondent groups (Ismail and Rahman, 2017). Also, Bruce and Hawes (2015), in their research related to a problem-solving intervention on 2D and 3D mental rotation by children (aged 4–8 years), found that all age groups demonstrated significant gains in their 2D mental rotation performance; the effects were higher and more consistent than those observed on the 3D tasks.

Figure 3 is an example of one of the 3D tasks, even though 3D tasks delivered lower success rates compared to 2D tasks (see Figure 1).

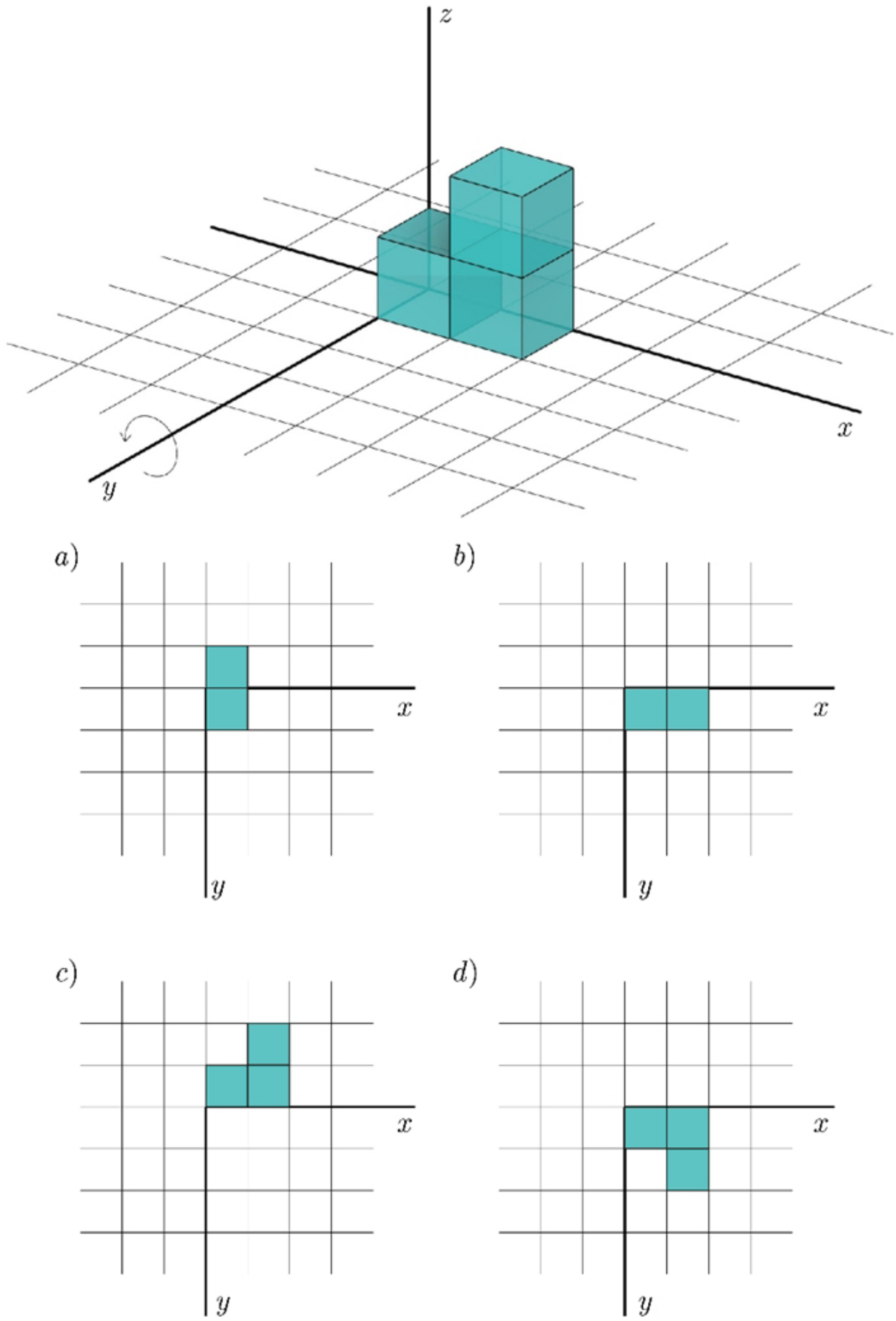

Figure 3: Task 13 – Identification of the projection of the object onto the *xy*-plane after rotating the object 270 degrees according to the indicated direction, 2021 (source: own drawing)

RQ4: How do deficiencies in conceptual knowledge contribute to students' errors in understanding tasks?

For Tasks 3, 5, 8 and 9, of which some of the subtasks were performed the least successfully, a more detailed analysis was undertaken of the possible causes of student errors due to insufficient conceptual knowledge. Our research confirmed that deficiencies in conceptual knowledge negatively affect student achievement.

In Subtasks 3.1 and 3.3 (see Figure 4), students determined the relative position of two straight lines in/on a cube based on a picture. Among the wrong answers, the following shortcomings often appeared in both tasks: instead of identifying the mutual positions of the given straight lines (parallel, intersecting, skew), the students used the formulation 'the straight lines intersect/do not intersect'. The most frequent error in Subtask 3.1 with two intersecting straight lines was the confusion between

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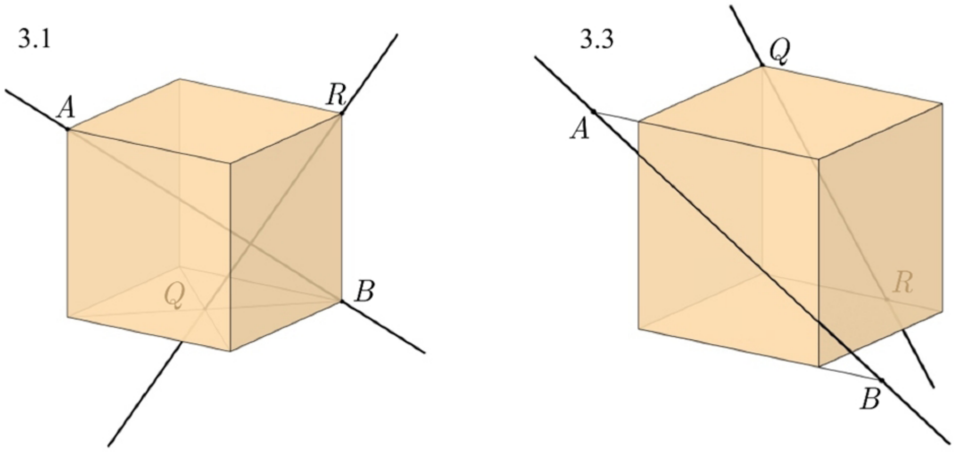

Figure 4: Subtask 3.1 and 3.3 – Identification of the relative position (i.e. parallel, intersecting, or skew) of two straight lines drawn in an auxiliary cube, 2021 (source: own drawing)

the metric relationship between the lines with a positional relationship: instead of intersecting lines, the students wrote about perpendicular straight lines. Other errors were mainly related to a lack of knowledge of the concept of skew and parallel straight lines, where, in both positions, the straight lines have an empty intersection, and it is necessary to distinguish whether the straight lines lie (for parallel ones) or do not lie (for skew ones) in the same plane. This was also indirectly confirmed by some respondents who stated in the notes to these tasks that they did not remember the necessary terms. Other studies have also identified that students have misconceptions about parallel straight lines (Biber et al., 2013; Youkap, 2021; Ulusoy, 2023) and perpendicular lines or segments (Duatepe-Paksu and Bayram, 2019; Ulusoy, 2023). This complies with Barut and Retnawati (2020), who have pointed to the problem of students' understanding of the concept of skew lines.

In each of the four subtasks of Task 5, the segment *AB* and its image *A'B'* under rotation around the given centre *S* by a certain oriented angle (the positions of *S* and *AB* are identical in all subtasks) were given. The situation was represented on a square grid. The task was closed. Students had to choose the appropriate size of the oriented angle from the offered options +90°, –90°, +60°, –60°, +45°, –45°. The correct answers were +90°, –90°, –60° and +45° in that order. We followed both the correct idea of the size of the angle of rotation, and the conceptual knowledge of rotation as such, especially the concept of the sense of rotation (plus/minus distinction). Students were the least successful in Subtasks 5.3 and 5.4. In the comments they mentioned a hesitation between 45° and 60°. However, they had greater difficulty with the plus/minus distinction, even though there was a hint in the assignment: 'plus corresponds to the sense of clockwise rotation'. If the absolute value was accepted as the correct answer (without the +/– distinction), only one answer would have been wrong in Subtasks 5.1 and 5.2, and the success rate would have increased noticeably in Subtasks 5.3 and 5.4 as well. Other researchers also draw attention to students' difficulties in determining the sense of rotation, for example (Clements et al., 1996; Clements and Burns, 2000). We consider the concept of the sense of rotation to be important for everyday life (e.g., opening a bottle, screwing, etc.).

Tasks with a low success rate also included Subtasks 8.3 and 9.1. In Tasks 8 and 9, students had to determine, without drawing pictures, the number of common points of two circles. Realising the relative position of the circles (Feng et al., 2014) was key to solving the tasks. Although this topic is not explicitly stated in the national curriculum (MEYS, 2005), it is included as a standard in the teaching of mathematics in lower secondary schools. The problem is usually visualised for students, and they solve the related problems using sketches. At the same time, they are led to derive the relationship between *s* (the distance between the centres of the given circles), r_1 and r_2 (the radii of the circles) for individual relative positions. In Task 8, the radii r_1, r_2 of both circles were given, and in the individual subtasks only the distance *s* was changed. In Task 9, only the radius of one circle was fixed, and the radius of the other and *s* varied. Many respondents found Tasks 8 and 9 difficult (as evidenced by their comments), especially because they were not allowed to use a picture, which is not usual in school teaching or in mathematics textbook tasks. Kambilombilo and Sakala (2015) also pointed out the difficulties with tasks beyond those common in textbooks.

Without a full understanding of a mathematical concept, the subsequent concept cannot be well understood (Aktaş and Ünlü, 2017; Hacisalihoğlu Karadeniz et al., 2017). Students need to have developed concepts such as a circle, intersection, point of tangency, etc. to understand the relative positions between two circles. For example, Hromadová et al. (2020) describe a misconception: 'the centre of a circle belongs to the circle'. This misconception can negatively affect the understanding of the relationship between two circles and the determination of the number of points they share in common. Some students in our test also stated that two circles have (exactly) three points in common.

Students with insufficient conceptual understanding rely on procedural understanding (Son, 2006). Tasks 8 and 9 would not be difficult without the prohibition of using an image. The low success rate points to difficulties in remembering and correctly visualising and comparing more data (three distances). In Subtasks 8.3 and 9.2, the circles shared one point in common (they touched internally). Through statistical analysis, it was found that the students' answers to these two subtasks are statistically dependent, i.e. that the students either solved both tasks or made mistakes in both. Subtasks 8.3 and 9.1 were among the tasks that were performed the least successfully, which is in line with our experience that the idea of circles touching internally and the related connections are the most difficult for students. The low success rate for Subtask 9.1, where the circles intersected at two points, was apparently caused by the use of symbolic notation in the assignment. This reason was also repeatedly mentioned by students in the comments. Selden and Selden (1995) pointed out that college students (including high school mathematics teachers) failed to consistently interpret informally written mathematical statements into equivalent formal statements. Similarly, Mutodi and Mosimege (2021: 1195) found that 'Misconceptions and poor conceptions in the interpretation of mathematical symbols result in students failing to link mathematical symbols and formulae with appropriate concepts'.

The above points to the fact that students' results in spatial ability may be related to students' conceptual knowledge. Other researchers also draw attention to this relationship in mathematics (e.g., Lowrie et al., 2019; Rittle-Johnson et al., 2019;) and also in other subjects, such as chemistry (Black, 2005).

Our findings show that conceptual misunderstandings contribute to students' errors in the performance of geometric tasks; however, we acknowledge that this is not the only possible explanation. In the case of 3D tasks, the lower success rates may also be due to the fact that less time is devoted to spatial geometry in the Czech mathematics curriculum. We cannot determine which factor contributes more to students' errors.

Limitations and Implications of the Study

Among the main limitations of our study is that it was conducted at a single institution and involved a relatively small sample of students, which may limit the generalisation of the results. Although the test was carefully designed to assess specific subcomponents of spatial ability, the classification of tasks into categories is partially based on expert judgement. Furthermore, some tasks may fall into multiple categories or may not be typical for a given category. We are also aware that the analysis should ideally include a qualitative component such as a more detailed analysis of students' written comments or additional comprehensive interviews. Future research may benefit from broader samples and additional qualitative data to further validate and expand our findings.

Despite these limitations, we believe that the results of our study offer important ideas for the design or modification of geometry courses at the university level. First, identifying specific subcomponents of spatial ability that present difficulties – particularly spatial visualisation – ensures that the instruction can include targeted practice with appropriate types of geometric problems. Second, examining students' conceptual misunderstandings provides meaningful feedback for improving the teaching of core geometry concepts, such as the relationships between lines and planes or between circles. Finally, because the study focuses on pre-service teachers, it has the potential to influence future teaching practice. These results can help improve the structure and content of university geometry courses to better prepare future mathematics teachers. This study may also serve as an inspiration for other researchers or educators aiming to improve spatial reasoning and conceptual understanding in geometry education.

CONCLUSION

The research presented in this paper provides insights into the spatial abilities of pre-service mathematics teachers in their first year of studies at the Faculty of Mathematics and Physics, Charles University, highlighting their varying proficiency across different geometric tasks.

Over three years of testing, it became evident that tasks involving spatial visualisation posed the most significant challenges, with students achieving a lower success rate compared to tasks requiring, for example, mental rotation. These findings underline the complexity of spatial visualisation tasks, which often require more difficult mental transformations than simple rotations. Furthermore, the research revealed a consistent trend: students performed better in planar geometric tasks compared to spatial ones, aligning with the broader educational focus on planar geometry within the Czech national curriculum. This outcome calls for enhanced training on spatial geometry, potentially improving educational outcomes in more complex 3D spatial reasoning tasks and for better support of the development of spatial abilities in pre-service mathematics teachers. The findings also emphasise the pivotal role of conceptual knowledge in understanding and solving geometric tasks. Throughout the research, it became clear that deficiencies in conceptual understanding significantly contributed to students' difficulties. For example, students struggled with tasks that required a deeper understanding of geometric properties and relationships. Improving conceptual knowledge can be considered a key strategy for enhancing students' overall performance in geometry. This is particularly important throughout the preparation of pre-service teachers, as they will be responsible for teaching these skills to future generations.

The findings are significant, as this is one of the first studies to use the same set of geometry tasks across three years to examine how first-year pre-service mathematics teachers perform in geometric problem solving. As for future research, the same group of students will be tested again at the end of their university studies. This will allow us to compare students' results at the beginning and end of their studies and to analyse whether the interventions provided through the university geometry course have been effective.

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